S13

Mathematics Support Centre

Title: Binomial Distribution

Target: On completion of this worksheet you should be able to identify key assumptions and calcuate probabilites of binomial distributions

Key Information to start:
Binomial distribution is broke down into trials giving probabilites.
The number of trials is how many times a situation is repeated.
Trials have to be independant to each other which means everytime you repeat the situation it has no effect on any other trial taking place now or in the future.
Each trial has 2 possible outcomes (success or failure) whose probabilities must remain constant from one trial to the other.

Key Notation
Number of trials $= n$
Number of successes $= m$ where $m = 0, 1, ..., n$
Probability of success $= p$
Probability of failure $= q$

where $p + q = 1$ or alternatively $q = 1 - p$
This is because all probabilites must add up to 1.
p and q must remain constant from one trial to the other.

General form
We write that, the number of successes m, follows a binomial distribution, $X \sim B(n, p)$
This is calculated by using the formula
$P(m) = {}^nC_m p^m q^{n-m}$

Key Assumptions
1) Number of trials are fixed
2) Only 2 possible outcomes for each trials
3) Each trial is independent of the others
4) The probability of each outcome remains constant from trial to trial

More Information
You may see that the coefficient of a probability is ${}^nC_m$.
This is spoken as the number of ways "n choose m" and you can calculate ${}^nC_m$ on the calculator by pressing SHIFT and then ÷. To calculate this manually we use the formula of $\dfrac{n!}{m!(n-m)!}$.

The mean of Binomial distribution is $\mu = np$
The variance of Binomial distribution is $\sigma^2 = npq$

Examples
Kate rolls an unbiased die 20 times. The die contains the numbers 1 to 6. Find the probability that:
1) Kate rolls a $5, 3$ times
2) Kate rolls less than $3, 10$ times
3) Kate rolls a 3 or less, 14 times

Answer

1) We know that:
no. of trials $n = 20$
probability of success $p = \frac{1}{6}$
probability of failure $q = 1 - \frac{1}{6} = \frac{5}{6}$

Since every number has an equal chance of being rolled. So
$X \sim B\Big(20, \frac{1}{6}\Big)$ can be calculated by

$$P(m) = \ {}^{n}C_m \ p^m \ q^{n-m} \text{ with}$$

$P(m = 3) = \ {}^{20}C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{17}$
$= 0.237887$

2) Now the probability has changed, we must change our conditions for the Binomial and therefore need to change X to another letter to represent the change from the previous Binomial conditions. We need to find the probability of rolling a 1 or 2. Since we need "or" we add the probabilities together, if it was "and" then we would multiply the fractions.

$\frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$
$p = \frac{1}{3}$, $q = \frac{2}{3}$ and $m = 10$
$Y \sim B\Big(20, \frac{1}{3}\Big)$ so,
$P(m = 10) = \ {}^{20}C_{10} \left(\frac{1}{3}\right)^{10} \left(\frac{2}{3}\right)^{10}$
$= 0.054259$

3) The probability of success is the probability of getting either a $1, 2$ or 3 which is half the numbers of the die, therefore the probability of success is $\frac{1}{2}$
$p = \frac{1}{2}$, $q = \frac{1}{2}$ and $m = 14$
$Z \sim B\Big(20, \frac{1}{2}\Big)$
$P(m = 14) = \ {}^{20}C_{14} \left(\frac{1}{2}\right)^{14} \left(\frac{1}{2}\right)^{6}$
$= 0.036964$

Biased die We have an obscurred die where the probability of throwing a 5 is $\frac{1}{3}$ while the remaining numbers to 6 is equal. Find the probability of throwing 5 at least 6 times out of 8 throws.

From this we know that the probability of throwing a 5 is $\frac{1}{3}$ so $p = \frac{1}{3}$ therefore $q = 1 - \frac{1}{3} = \frac{2}{3}$

$n = 8$ and $m = 6$
At least means that the lowest we can throw is the minimal number but we are able to throw more than our minimal number and we must calculate their probilities, however since it's only being thrown 8 times once, they would be "or" probabilities, which means we add their probabilities together to find our answer. So we need
$P(x \geq 6) = P(x = 6) + P(x = 7) + P(x = 8)$
Using $X \sim B\Big(8, \frac{1}{3}\Big)$ we get

$${}^{8}C_6 \frac{1}{3}^6 \frac{2}{3}^2 + {}^{8}C_7 \frac{1}{3}^7 \frac{2}{3}^1 + {}^{8}C_8 \frac{1}{3}^8 \frac{2}{3}^0 = 0.02 \ (to \ 2dp).$$

So the probability of throwing a 5 on this biased die at least 6 times out of 8 throws is 0.02

Exercises

1. Find the probability that in throwing a fair coin 3 times, there will be
   a) 3 heads
   b) 2 tails
   c) at least 1 head
   d) no more than 1 tail

2. If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen at random the following amount of bolts are defective:
   a) 1
   b) 0
   c) less than 2

3. If the probability of defective bolt is 0.1 in a collection of 400 bolts, find
   a) the mean
   b) the standard deviation

4. Suppose that 80% of adults with allergies report symptomatic relief with a specific medication. If the medication is given to 10 new patients with allergies, what is the probability that it is effective in exactly seven?

5. You are performing a cohort study. If the probability of developing disease in the exposed group is 0.05 for the study duration, then if you sample (randomly) 500 exposed people, how many do you expect to develop the disease? Give a margin of error (+/- 1 standard deviation) for your estimate.

6. We have a biased die where the probability of throwing a 4 is $\frac{1}{4}$ with the remaining numbers $1, 2, 3, 5$ and 6 are equal. Out of 6 throws find the probability of throwing a 2 is at most 1 time.

Answers

1. a) $\frac{1}{8}$
   b) $\frac{3}{8}$
   c) $\frac{7}{8}$
   d) $\frac{1}{2}$

2. a) 0.4096
   b) 0.4096
   c) 0.8192

3. a) $\mu = np = 40$ expect 40 defective bolts
   b) $\sigma^2 = npq = 36 \quad \sigma = 6$

4. 0.2013

5. 25 ± 4.87

6. $\frac{1 - 0.25}{5} = 0.15$
   $X \sim B\big(6, 0.15\big)$
   $P(x \leq 1) = P(x = 0) + P(x = 1)$
   $= 0.78$ (*to 2dp*)