Mathematics

Support Centre

# Title: Solving Ordinary Differential Equations (ODE's)

**Target:** On completion of this workbook you should be able to recognise and apply the appropriate method for solving a given ordinary differential equation.

## Contents


**1 Identifying your Ordinary Differential Equation** 2

**2 Introduction** 3

**3 Separation Of Variables** 5

**4 Change Of Variable** 6

**5 Equations That Are Reducible To Homogeneous Form** 7

**6 Integrating Factors** 9

**7 Bernoulli Equations** 10

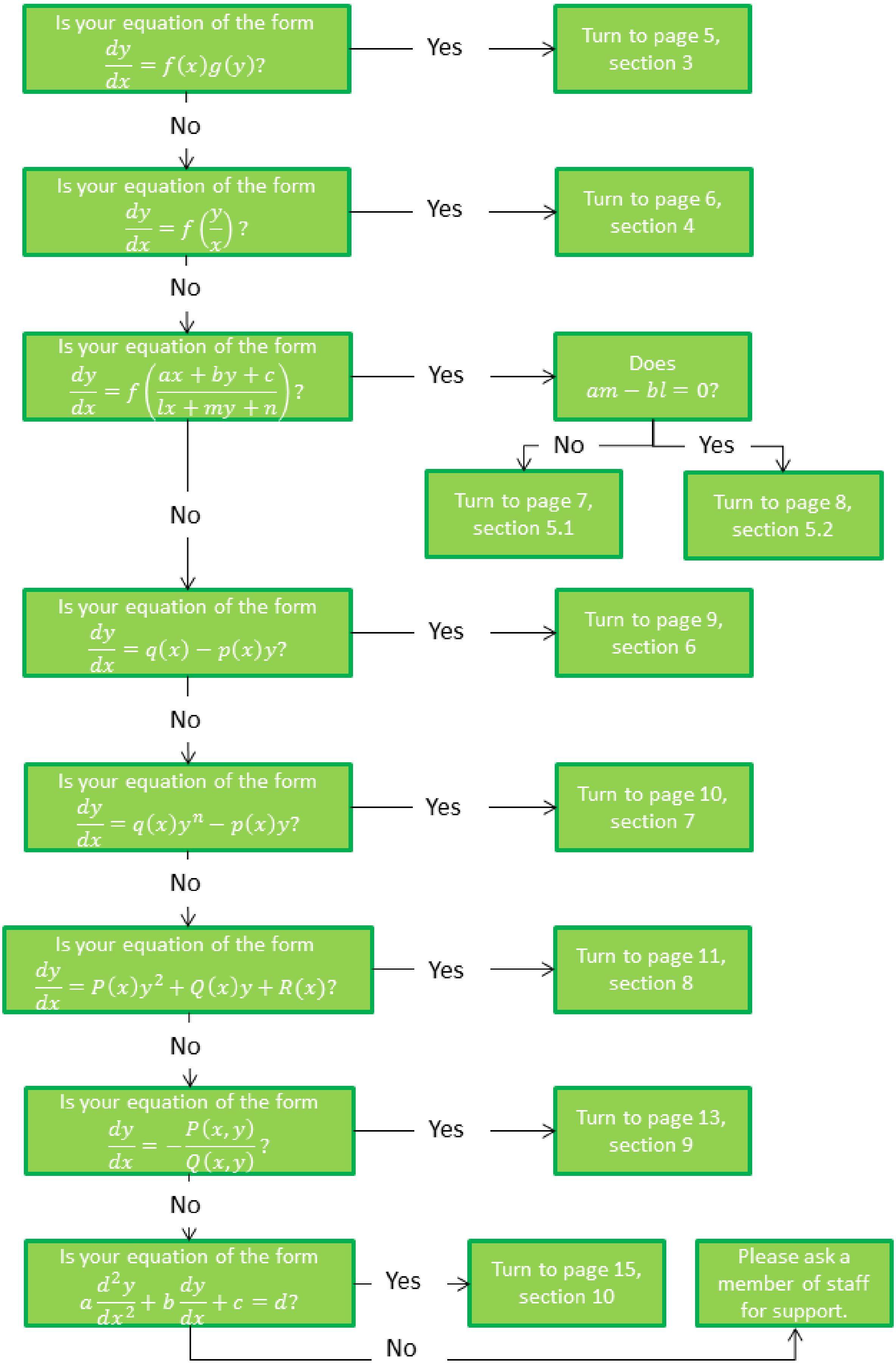
**8 Riccati Equations** 11

**9 Exact Equations** 13

**10 Second-Order Linear Differential Equations with Constant Coefficients** 15

**11 Solutions** 22

# 1 Identifying your Ordinary Differential Equation



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# 2 Introduction

**How to write derivatives.** Let $y$ be a function of an independent variable $x$, usually written as $y = f(x)$. The (first) derivative of $y$ with respect to $x$ is denoted by $\frac{dy}{dx}$. It can also be written $y'$ or $f'(x)$ using the prime notation invented by Joseph Lagrange† . The second and, more generally, the $n$th derivatives are written like this

$$\frac{d^2y}{dx^2},\ y'' \text{ or } f''(x), \quad \cdots, \quad \frac{d^ny}{dx^n},\ y^{(n)} \text{ or } f^{(n)}(x).$$

We may use Lagrange's prime notation because it is more compact and doesn't change the spacing between lines with in-line formulas.

**What is a Differential Equation?** The most general form of differential equation involves sums of powers of various derivatives with coefficients that are functions of the independent variable. Here is a hairy example:

$$(1+x^2)\left(\frac{d^5y}{dx^5}\right)^2 - 3\sin x\left(\frac{d^2y}{dx^2}\right)^3 - 99\frac{dy}{dx} + e^xy = \tan(5x^4 - 2\log x) \tag{1}$$

Don't worry, you won't meet any equations as nasty as this one. Almost any differential equation that you could imagine like this can't be solved in 'closed form'†. The good news is that we only study a few special types of ODEs that *can*.

**Why do we call them 'ordinary'?** The reason is to highlight the fact that we are only dealing here with functions of a *single* independent variable, such as $y$ is a function of $x$, or $W = W(u)$, a function of $u$, or $\theta = \theta(t)$, and so on. To find the derivatives of functions of more than one variable, such as

$$f(x, y) = x^3 + 2xy - 3y^2,$$

we differentiate one variable at a time and call them *partial* derivatives. In the given example, we can differentiate the function $f$ partially with respect to $x$ (regarding $y$ as a constant) and partially with respect to $y$ (regarding $x$ as a constant) as follows:

$$\frac{\partial f}{\partial x} = 3x + 2y \quad \text{and} \quad \frac{\partial f}{\partial y} = 2x - 6y.$$

Notice that the Roman letter $d$ in an ordinary derivative has been replaced with a stylised version† $\partial$ to show that more than one independent variable is involved.

### The 'Order' and the 'Degree' of a differential equation.

The *order* is the largest integer $n$ such that $\frac{d^ny}{dx^n}$ appears in the equation (with non-zero coefficient!). Thus equation 1 has order 5. The equations we study here have orders 1 or 2.

The *degree* is the largest power of a derivative appearing in the equation. In equation 1 the second derivative $\frac{d^2y}{dx^2}$ appears to the third power and so it has degree 3. All the types of equations we meet here have degree 1.

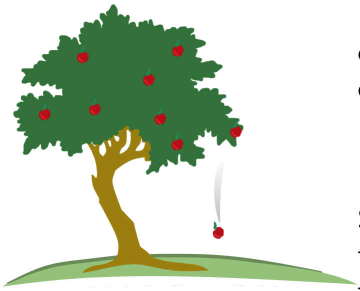
†A famous French mathematician who lived from 1736 to 1813.

†*Closed form* means a mathematical expression containing constants, variables, certain "well-known" operations (e.g. addition, subtraction, multiplication and division) and standard functions (such as $n$th roots, powers, logarithmic and exponential functions, trigonometric and hyperbolic functions, and their inverses),

†The notation was introduced by another distinguished French mathematician, Adrien-Marie Legendre, in 1786.

**What is the point of differential equations?** They enable us to study change in the real world. For instance, if the position of an object is determined by a distance $s$ which changes with time $t$ according to some function $s = s(t)$, then

$$\frac{ds}{dt} \text{ will be the \textit{speed} of the object and } \frac{d^2s}{dt^2} \text{ its \textit{acceleration}.}$$



Isaac Newton's *second law of motion* says that if the combined forces on an object of mass $m$ are denoted by $F$, then the acceleration $a$ of the object (in the direction of the force) is determined by the equation

$$F = ma \tag{2}$$

Suppose $s$ is the distance of a stone from the ground and that the stone is allowed to fall under the force $g$ of gravity. Newton's *law of universal gravitation* implies that the force on the stone is $mg$ vertically downward. (negative) direction. If we ignore air resistance, equation 2 tells us that

$$-mg = ma = m\frac{d^2s}{dt^2}, \quad \text{and so cancelling the } m\text{'s, we get the differential equation } \frac{d^2s}{dt^2} + g = 0$$

Since $g$ is constant, this equation has a family of solutions $s = A + Bt - gt^2/2$ where the initial conditions determine the values of the constants $A$ and B . If the stone is dropped from rest at time $t = 0$ from a height $s = h$ above the ground, the values are $A = h$ and $B = 0$; therefore the equation of motion of the stone is

$$s = h - \frac{1}{2}gt^2. \tag{3}$$

*Two comments:* 1. Observe that equation 3 does not involve the mass $m$ of the object. This justifies the claim made by the Italian astronomer, physicist, engineer, philosopher and mathematician, Galileo Galilei (1564 – 1642), that the time taken for a dropped object to fall to the ground does not depend on its weight. The celebrated story that he proved his claim by dropping two stones of differing weights from the top of the leaning tower in his home town of Pisa, is engaging but probably apocryphal.

2. By setting $s = 0$, we obtain the length of time it takes for the stone to hit the ground; this is equal to $\sqrt{2h/g}$.

Many of the great leaps forward in scientific understanding of our universe have been driven by elegant differential equations. The equations of Newtonian mechanics and Einstein's relativity allow us to send satellites into orbit round the tiny comet 67P/Churyumov-Gerasimenko 186 million kilometres from earth.

- The atom bomb and radiocarbon dating are both products of the differential equation $\boldsymbol{y' = ky}$. This equation defines the exponential function $y = e^{kx}$. When $k > 0$, you get exponential *growth* and a big explosion. On the other hand, when $k < 0$ the equation models radioactive *decay*, for instance.
- Option pricing in financial markets is described by the famous **Black-Scholes Differential Equation**. This can be solved by transforming it into the long-studied and well-understood **Heat Equation** which models thermal conductivity and diffusion, as well as Brownian motion in probability theory.
- Weather-forecasting and keeping 400 tons of jumbo jet aloft depend heavily on the **Navier-Stokes Equation** which describes viscous flow in liquids and gases.
- Your TV and your smart phone would not exist without **Maxwell's Equations**, which encapsulate the astonishing properties of electricity and magnetism.
- Better time-keeping and microscopy, unbreakable codes, our knowledge of chemical and biological processes are all possible through our understanding the mysteries of the quantum world revealed by **Schrödinger's Equation**.
- Whether an outbreak of measles will become an epidemic is accurately determined by the elementary simultaneous **SIR Differential Equations**, They also tell us what proportion of the population need to be vaccinated to avoid one.

## 3 Separation Of Variables

**For an ODE in the form: $\frac{dy}{dx} = f(x)g(y)$.**

The solution is found by using the method known as 'separation of variables'.

Essentially, this is the process of gathering all $x$ components on one side of the equation, and all $y$ components on the other. Following this, both sides may be integrated directly to solve the problem.

**Worked example**

Suppose that we have the following ODE:

$$\frac{dy}{dx} = \frac{x}{y+2}$$

The solution is found by separating the variables and integrating both sides. In this instance, this is done by multiplying by $dx$ and $y + 2$. This gives:

$$(y + 2)dy = xdx$$

We may now integrate both sides:

$$\int (y + 2)dy = \int xdx$$

This gives:

$$\frac{y^2}{2} + 2y = \frac{x^2}{2} + C$$

Where $C$ is some arbitrary constant of integration.

We note at this stage that if we were given an initial condition, i.e. $y(0) = 1$, we may find the value of $C$, thus giving us a particular solution.

**Exercise**

Solve the following with separation of variables:

1. $\frac{dy}{dx} = \frac{y^2}{x^2}$
2. $\frac{dy}{dx} = \frac{2x + 3}{y}$

# 4 Change Of Variable

**For an ODE in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.**

The equation is homogeneous and the solution is found by 'changing the variable' and using the substitution $v = \dfrac{y}{x}$, and then solving the resulting separable equation.

**Worked Example**

Suppose that we have the following ODE:

$$\frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

We consider the substitution $y(x) = xv(x)$, i.e. $v = \frac{y}{x}$.
By the product rule, $\frac{dy}{dx} = x\frac{dv}{dx} + v$.

Therefore, the equation becomes:

$$x\frac{dv}{dx} + v = v - v^2$$

Hence,

$$x\frac{dv}{dx} = -v^2$$

This equation is now separable and may be solved in the following way:

$$\int -\frac{1}{v^2}dv = \int \frac{1}{x}dx$$

Therefore,

$$\begin{aligned}\frac{1}{v} &= \ln(x) + \ln(k)\\ &= \ln(kx)\end{aligned}$$

Where $\ln(k)$ is an arbitrary constant of integration. By substituting $v = \frac{y}{x}$ and simplifying, we obtain:

$$\frac{x}{y} = \ln(kx)$$

$$y = \frac{x}{\ln(kx)}$$

**Exercise**

Solve the following using a change of variable:

1. $\dfrac{dy}{dx} = \dfrac{\frac{3y}{x} - 1}{3 - \frac{y}{x}}$
2. $\dfrac{dy}{dx} = \dfrac{y^2}{x^2} + \dfrac{y}{x} + 1$

# 5 Equations That Are Reducible To Homogeneous Form

## 5.1 Case 1: $am - bl \neq 0$

**For an ODE in the form: $\frac{dy}{dx} = f\left(\frac{ax+by+c}{lx+my+n}\right)$. Where $am - bl \neq 0$.**

This equation would be homogeneous if the terms $c$ and $n$ equalled $0$. However if they do not equal $0$, we need to reduce them to homogeneous form.
By using the substitutions $x = X + \alpha$ and $y = Y + \beta$, it is possible to reduce these terms to zero and hence solve the equation. Essentially, we shall use:

$$ax + by + c = aX + bY + (c + a\alpha + b\beta)$$
$$lx + my + n = lX + mY + (n + l\alpha + m\beta)$$

Provided $am - bl \neq 0$, it is possible to choose $\alpha$ and $\beta$ such that $c + a\alpha + b\beta = 0$ and $n + l\alpha + m\beta = 0$ and the differential equation will then take the form:

$$\frac{dY}{dX} = f\left(\frac{aX+bY}{lX+mY}\right)$$

Which is a homogeneous equation that can be solved.

**Worked example**

Suppose that we have the following ODE: $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$

First, put $x = X + \alpha$ and $y = Y + \beta$. Then,

$$y - x + 1 = Y - X + (\beta - \alpha + 1)$$
$$y + x + 5 = Y + X + (\beta + \alpha + 5)$$

It can then be seen that $\beta - \alpha + 1 = 0$ and $\beta + \alpha + 5 = 0$ if $\alpha = -2$ and $\beta = -3$. With this choice, the differential equation becomes:

$$\frac{dY}{dX} = \frac{Y-X}{Y+X}$$

We now use the substitution $\frac{Y}{X} = v$ so that $\frac{dY}{dX} = v + X\frac{dv}{dX}$ (see section 2). From this then, we obtain:

$$v + X\frac{dv}{dX} = \frac{Xv - X}{Xv + X} \Rightarrow v + X\frac{dv}{dX} = \frac{v-1}{v+1} \Rightarrow X\frac{dv}{dX} = \frac{-v^2-1}{v+1}$$

Separating the variables gives:

$$\int \frac{1}{X}dX = -\int \frac{v+1}{v^2+1}dv \Rightarrow \int \frac{1}{X}dX = -\int \frac{v}{v^2+1} + \frac{1}{v^2+1}dv$$

Hence,

$$\ln(X) + \ln(C) = -\frac{1}{2}\ln(v^2+1) - \tan^{-1}(v)$$
$$\ln(X^2) + \ln(v^2+1) = \ln(C^2) - 2\tan^{-1}(v)$$

Where C is some arbitrary constant of integration.

With some rearrangement, this gives: $X^2(v^2+1) = C^2 e^{(-2\tan^{-1}(v))}$

Recalling our substitution, we obtain: $X^2\left(\frac{Y^2}{X^2}+1\right) = C^2 e^{\left(-2\tan^{-1}\left(\frac{Y}{X}\right)\right)}$

As $X = x + 2$ and $Y = y + 3$, our final answer may be given by: $(x+2)^2 + (y+3)^2 = C^2 e^{\left(-2\tan^{-1}\left(\frac{y+3}{x+2}\right)\right)}$

### 5.2 Case 2: $am - bl = 0$

**For an ODE in the form: $\frac{dy}{dx} = f\left(\frac{ax+by+c}{lx+my+n}\right)$. Where $am - bl = 0$.**

In the above example, we assumed that $am - bl \neq 0$. If this is not the case, i.e. $am - bl = 0$, the equation

$$\frac{dy}{dx} = f\left(\frac{ax+by+c}{lx+my+n}\right)$$

may be solved by setting $ax + by = z$.

**Worked example**

Suppose that we have the following ODE: $\frac{dy}{dx} = \frac{x-y+1}{2x-2y+3}$

In this instance, $am - bl = (1 \times -2) - (-1 \times 2) = 0$.

We proceed then by putting $z = ax + by = x - y$.

This also means that $\frac{dy}{dx} = 1 - \frac{dz}{dx}$.

The ODE above therefore becomes:

$$1 - \frac{dz}{dx} = \frac{z+1}{2z+3} \Rightarrow \frac{dz}{dx} = \frac{z+2}{2z+3}$$

This may now be solved by separating variables.

$$\int \frac{2z+3}{z+2} dz = \int dx$$

This may be written as:

$$\int \frac{2z+4-1}{z+2} dz = \int dx$$

Which simplifies to:

$$\int \frac{2(z+2)-1}{z+2} dz = \int dx$$

$$\int 2 - \frac{1}{z+2} dz = \int dx$$

$2z - \ln(z+2) = x + C$ Where $C$ is an arbitrary constant of integration.

Recalling our substitution $z = x - y$, this gives:

$$2(x-y) - \ln(x-y+2) = x + C$$

Or,

$$x - 2y - \ln(x-y+2) = C$$

**Exercise**

Solve the following:

1. $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$
2. $\frac{dy}{dx} = \frac{2x+3y+4}{4x+6y+5}$

# 6 Integrating Factors

**For an ODE in the form: $\frac{dy}{dx} = q(x) - p(x)y$.**

By multiplying both sides of the differential equation by the integrating factor, $I$, which is defined as $I = e^{(\int p(x)dx)}$, it is possible to solve the equation.

**Worked example**

Suppose that we have the following ODE: $\frac{dy}{dx} = x^2 - \left(\frac{2}{x}\right) y$

The integrating factor for this equation is given by:

$$I = e^{\left(\int \frac{2}{x} dx\right)}$$

Hence,

$$I = e^{(2\ln(x))} = e^{\left(\ln(x^2)\right)} = x^2$$

We now multiply the entire ODE by $I$ to give:

$$x^2 \frac{dy}{dx} = x^4 - 2xy$$

$$x^2 \frac{dy}{dx} + 2xy = x^4$$

At this stage, we identify that by the product rule, this is the same as:

$$\frac{d}{dx}\left(x^2 y\right) = x^4$$

Which integrates to give:
$x^2 y = \frac{x^5}{5} + C$ Where $C$ is an arbitrary constant of integration.

Thus the general solution is given by:

$$y = \frac{x^3}{5} + \frac{C}{x^2}$$

**Exercise**

Solve the following using an integrating factor:

1. $\frac{dy}{dx} = \frac{1}{x} e^{(x)} - \left(1 + \frac{1}{x}\right) y$
2. $\frac{dy}{dx} = \frac{e^{(x)}}{x^3} - \left(\frac{3}{x}\right) y$

# 7 Bernoulli Equations

**For an ODE of the form: $\frac{dy}{dx} = q(x)y^n - p(x)y$.**

This is known as a Bernoulli equation. It can be simplified by letting $z = y^{1-n}$, which means $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$. Using this, we may obtain a linear equation and solve it by using an integrating factor (see section 4).

**Worked example**

Suppose that we have the following ODE: $\frac{dy}{dx} = xy^3 + 2xy$

We now let $z = y^{-2}$ and find that:

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{y^3}{2}\frac{dz}{dx}$$

Hence,

$$-\frac{y^3}{2}\frac{dz}{dx} = xy^3 + 2xy \Rightarrow \frac{dz}{dx} = -2x - 4xy^{-2}$$

Recalling our substitution, $z = y^{-2}$, we find that this gives:

$$\frac{dz}{dx} = -2x - 4xz$$

As this equation is linear, we now use the integrating factor $I = e^{(\int 4x dx)} = e^{(2x^2)}$. Multiplying the entire equation by $I$ gives the following result:

$$e^{(2x^2)}\frac{dz}{dx} = -2xe^{(2x^2)} - 4xe^{(2x^2)}z$$

Hence,

$$e^{(2x^2)}\frac{dz}{dx} + 4xe^{(2x^2)}z = -2xe^{(2x^2)}$$

Which may be written as:

$$\frac{d}{dx}\left(e^{(2x^2)}z\right) = -2xe^{(2x^2)}$$

This integrates to give:

$$e^{(2x^2)}z = -\frac{1}{2}e^{(2x^2)} + C$$

(Where $C$ is some arbitrary constant of integration).

$$z = -\frac{1}{2} + Ce^{(-2x^2)}$$

Recalling the substitution $z = y^{-2}$, this ultimately rearranges to give:

$$y = \frac{1}{\sqrt{Ce^{(-2x^2)} - \frac{1}{2}}}$$

**Exercise**

Solve the following Bernoulli equations:

1. $\frac{dy}{dx} = y^{1/2} - \frac{y}{x}$
2. $\frac{dy}{dx} = x^3y^2 - \frac{4}{x}y$

# 8 Riccati Equations

**For an ODE of the form:** $\boldsymbol{\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)}$**.**

This is known as a Riccati equation. Interestingly, when $P = 0$, the equation is linear. Also, when $R = 0$, it is a Bernoulli equation (see section 5).
A general solution may be obtained for this equation if a particular solution is first known.
Suppose that $y = v(x)$ is a known particular solution of the ODE, i.e. it satisfies the equation:

$$v' = Pv^2 + Qv + R$$

If this is satisfied, we consider the function $y = v(x) + \frac{1}{z(x)}$, this means that $y' = v' - \frac{z'}{z^2}$
Substituting the above into the ODE gives:

$$v' - \frac{z'}{z^2} = P\left(v^2 + \frac{2v}{z} + \frac{1}{z^2}\right) + Q\left(v + \frac{1}{z}\right) + R$$

As previously stated however, $v' = Pv^2 + Qv + R$. Hence:

$$-\frac{z'}{z^2} = P\left(\frac{2v}{z} + \frac{1}{z^2}\right) + Q\left(\frac{1}{z}\right)$$

Or,

$$z' + (2Pv + Q)\,z + P = 0$$

This is a linear equation whose general solution can be obtained.

**Worked example**

Suppose that we have the following ODE: $\frac{dy}{dx} = y^2 + \frac{y}{x} - \frac{3}{x^2}$

Upon inspection, it can be seen that $y = \frac{1}{x}$ satisfies the equation. We see this as:

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{x^2} + \frac{1}{x^2} - \frac{3}{x^2}$$

$$-\frac{1}{x^2} = -\frac{1}{x^2}$$

We use the substitution $y = \frac{1}{x} + \frac{1}{z}$ to obtain:

$$-\frac{1}{x^2} - \frac{z'}{z^2} = \left(\frac{1}{x^2} + \frac{2}{xz} + \frac{1}{z^2}\right) + \left(\frac{1}{x^2} + \frac{1}{xz}\right) - \frac{3}{x^2}$$

Which rearranges to give:

$$-\frac{z'}{z^2} = \left(\frac{2}{xz} + \frac{1}{z^2}\right) + \left(\frac{1}{xz}\right)$$

Hence,

$$z' + \frac{3z}{x} = -1$$

This is a linear equation that may be solved by using an integrating factor (see section 4).

We use the integrating factor $I = e^{\left(\int \frac{3}{x}dx\right)} = e^{(3\ln x)} = e^{\left(\ln x^3\right)} = x^3$



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Multiplying the entire differential equation by this gives:

$$x^3 z' + 3x^2 z = -x^3$$

This may be rewritten as:

$$\frac{d}{dx}\left(x^3 z\right) = -x^3$$

This integrates to give:

$$x^3 z = -\frac{x^4}{4} + C$$

Where $C$ is an arbitrary constant of integration.

This can be rearranged to give:

$$z = \frac{C}{x^3} - \frac{x}{4}$$

At this stage, we recall the substitution

$$y = \frac{1}{x} + \frac{1}{z} \Rightarrow z = \frac{1}{y - \frac{1}{x}}$$

Hence,

$$\frac{1}{y - \frac{1}{x}} = \frac{C}{x^3} - \frac{x}{4}$$

This rearranges to give the following:

$$1 = \left(y - \frac{1}{x}\right)\left(\frac{C}{x^3} - \frac{x}{4}\right)$$

$$y = \frac{1}{\frac{C}{x^3} - \frac{x}{4}} + \frac{1}{x}$$

Which finally gives:

$$y = \frac{4x^3}{C - x^4} + \frac{1}{x}$$

**Exercise**

Solve the following Riccati equations:

1. $\dfrac{dy}{dx} = y^2 - y - 2$
2. $\dfrac{dy}{dx} = y^2 - \dfrac{y}{x} - \dfrac{1}{x^2}$

# 9 Exact Equations

**For an ODE of the form: $\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}$.**

This equation can be solved as an exact equation if there exists a function $\phi(x, y)$ with $P(x, y) = \frac{\partial \phi}{\partial x}$ and $Q(x, y) = \frac{\partial \phi}{\partial y}$. The function $\phi$ may be found by solving these equations.

It is a necessary condition then, that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Once $\phi$ is found, the solution of the differential equation is given by $\phi = C$.

We also note that if $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, an exact equation can sometimes still be obtained by multiplying by a suitable function.

**Worked example**

Suppose that we have the following ODE: $\frac{dy}{dx} = \frac{2x^2 - 2y + y^2}{x(1-y)}$

This may be rewritten as $2x^2 - 2y + y^2 - x(1-y)\frac{dy}{dx} = 0$

This is of the form $P(x, y) + Q(x, y)\frac{dy}{dx} = 0$

We see that $P(x, y) = 2x^2 - 2y + y^2$, and $Q(x, y) = -x(1 - y)$

This is an exact equation if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$$\frac{\partial P}{\partial y} = -2 + 2y$$

$$\frac{\partial Q}{\partial x} = -1 + y$$

Clearly, $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, thus it is not an exact equation.

However, multiplying the entire equation by $2x$ gives:

$$4x^3 - 4xy + 2xy^2 - 2x^2(1-y)\frac{dy}{dx} = 0$$

We see from this, that $P(x, y) = 4x^3 - 4xy + 2xy^2$, and $Q(x, y) = -2x^2(1 - y)$

This leads us to:

$$\frac{\partial P}{\partial y} = -4x + 4xy$$

$$\frac{\partial Q}{\partial x} = -4x + 4xy$$

Which satisfies the condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Therefore, there must exist a function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = P = 4x^3 - 4xy + 2xy^2$$

$$\frac{\partial \phi}{\partial y} = Q = -2x^2(1-y)$$

Integrating these with respect to $x$ and $y$ respectively gives:

$$\frac{\partial \phi}{\partial x} = 4x^3 - 4xy + 2xy^2 \Rightarrow \phi = x^4 - 2x^2y + x^2y^2 + f(y)$$

$$\frac{\partial \phi}{\partial y} = -2x^2(1-y) \Rightarrow \phi = -2x^2y + x^2y^2 + g(x)$$

Where $f(y)$ and $g(x)$ are simply unknown functions of $y$ and $x$ respectively.

Comparing the above, these are consistent if $\phi = x^4 - 2x^2y + x^2y^2 + C$.

This means that $g(x) = x^4$ and $y(x) = C$, where $C$ is some arbitrary constant.

Hence, the differential equation has the solution: $x^4 - 2x^2y + x^2y^2 + C = 0$

**Exercise**

Solve the following:

1. $2x\cos(2y) + 1 + (4y^3 - 2x^2\sin(2y))\dfrac{dy}{dx} = 0$
2. $2xy - 9x^2 + (2y + x^2 + 1)\dfrac{dy}{dx} = 0$

# 10 Second-Order Linear Differential Equations with Constant Coefficients

A second-order linear differential equation, looks like this:

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x) \tag{1}$$

Here $y = y(x)$, a function of an independent variable $x$.
If the coefficient functions are constant: $a(x) = a,\ b(x) = b,\ c(x) = c$, we have an equation of the type described in the title; this equation can then be written more tidily using the prime notation† like this:

$$ay'' + by' + cy = f(x) \tag{2}$$

**Our Solving Strategy**. There are three steps.

*Step 1:* We first solve the so-called *homogeneous equation* obtained by setting $f(x) = 0$. Our solution will have different forms depending on the roots of the *auxiliary polynomial* (the quadratic equation $at^2 + bt + c = 0$) and will include two arbitrary constants; these constants can be determined if we are given initial (or boundary) conditions. The solution of the homogeneous equation is called *the complementary function*, abbreviated to CF. Because of the arbitrary constants, the complementary function is in fact a whole family of solutions to the homogeneous equation.

*Step 2:* Next we find a *particular integral* (PI). This is just one particular function $y(x)$ that satisfies equation 1. For a few special types of function $f(x)$ we will describe a systematic way to find a particular integral.

*Step 3:* We add a particular integral to the complementary function to get the *general solution* (GS); thus GS = CF + PI and *every* solution will have this form. This is because if $y_1(x)$ and $y_2(x)$ satisfy equation 2, their difference $y_1(x) - y_2(x)$ satisfies the homogeneous equation. Therefore any two solutions differ by an instance of the complementary function.

## 10.1 Solving the Homogeneous Equation

**For an ODE in the form: $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$**

Recall that $a$,$b$ and $c$ in this equation are constants (real numbers). We will see that the solution of this homogeneous equation depends on the roots of the *auxiliary equation*, the quadratic

$$at^2 + bt + c = 0. \tag{1}$$

We consider a trial solution in the form $y = e^{mx}$. Differentiating twice with respect to x:

$$y' = me^{mx}$$
$$y'' = m^2e^{mx}$$

and now substituting these values of $y, y'y''$ back into the original differential equation, we get:

$$\left(am^2 + bm + c\right)e^{mx} = 0$$

Since the exponential function $e^{mx}$ is never zero, it follows $am^2 + bm + c = 0$, and so $t = m$ is a root of the auxiliary equation 1. By the well-known formula for the roots of a quadratic equation, we obtain two possible solutions for $m$:

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
$$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

This then gives two possible solutions of the homogeneous equation: $y_1(x) = e^{m_1x}$ and $y_2(x) = e^{m_2x}$.

†The prime notation for derivatives was introduced by the French mathematician Joseph Lagrange (1736-1813).

It is important to distinguish the three cases that depend on the coefficients of the auxiliary equation.

*Case 1:* $b^2 > 4ac$ when two roots $m_1$, $m_2$ are real and distinct (i.e. $m_1 \neq m_2$).
*Case 2:* $b^2 = 4ac$ when the equation has a repeated root $m$.
*Case 3:* $b^2 < 4ac$ when the two roots $m_1$, $m_2$ are distinct conjugate complex numbers.

The corresponding solutions of the homogeneous equation, namely the complementary functions, are given in the following table:

| Case | Complementary Function |
|---|---|
| $b^2 > 4ac$ | $y = Ae^{m_1 x} + Be^{m_2 x}$ |
| $b^2 = 4ac$ | $y = (A + Bx)e^{mx}$ |
| $b^2 < 4ac$ | Let $m = \alpha \pm i\beta$. Then $y = (A\cos(\beta x) + B\sin(\beta x))e^{\alpha x}$ |

**Worked example. Case 1: $b^2 > 4ac$**

Suppose that we have the following ODE: $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + x = 0.$

Note that the form of the equation indicates that $x$ is to be seen as a function of an independent variable $t$ (perhaps denoting time), and also that, since the equation is already homogeneous, the complementary function will be the general solution. Now the roots of the auxiliary equation

$$m^2 - 3m + 1 = 0,$$

may be given by the quadratic formula,

$$m = \frac{3 \pm \sqrt{9-4}}{2} = \frac{1}{2}(3 \pm \sqrt{5}).$$

The general solution is therefore

$$x = Ae^{\frac{1}{2}(3+\sqrt{5})t} + Be^{\frac{1}{2}(3-\sqrt{5})t}. \tag{2}$$

Suppose however, that we are given the initial conditions

$$x = 1 \text{ and } \frac{dx}{dt} = 0 \text{ when } t = 0.$$

We will now consider how to find the particular solution satisfying these conditions.
First, we substitute $t = 0$ and $x = 1$ into equation 2. Since $e^0 = 1$, we see that

$$1 = A + B \tag{3}$$

Next, we differentiate equation 2 with respect to $t$

$$\frac{dx}{dt} = \frac{1}{2}(3+\sqrt{5})Ae^{\frac{1}{2}(3+\sqrt{5})t} + \frac{1}{2}(3-\sqrt{5})Be^{\frac{1}{2}(3-\sqrt{5})t}$$

and set $t = 0 = \dfrac{dx}{dt}$ in this equation to obtain

$$0 = \frac{1}{2}(3+\sqrt{5})A + \frac{1}{2}(3-\sqrt{5})B$$

Hence

$$0 = (3+\sqrt{5})A + (3-\sqrt{5})B = 3(A+B) + \sqrt{5}(A-B) \tag{4}$$

From equation 3 we can substitute $B = 1 - A$ in equation 4 to obtain

$$0 = 3 + \sqrt{5}(2A - 1)$$

From this it follows easily that

$$A = \frac{1}{2}\left(1 - \frac{3}{\sqrt{5}}\right) B = \frac{1}{2}\left(1 + \frac{3}{\sqrt{5}}\right).$$

The particular solution to the original homogeneous equation satisfying the given boundary conditions is therefore

$$x = \frac{1}{2}\left(1 - \frac{3}{\sqrt{5}}\right) e^{\frac{1}{2}(3+\sqrt{5})} + \frac{1}{2}\left(1 + \frac{3}{\sqrt{5}}\right) e^{\frac{1}{2}(3-\sqrt{5})}.$$

**Worked example. Case 2: $b^2 = 4ac$**

Suppose that we have the following ODE: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

Note that the form of the equation indicates that $y$ is to be seen as a function of an independent variable $x$, and also that, since the equation is already homogeneous, the complementary function will be the general solution. Now the roots of the auxiliary equation

$$m^2 - 4m + 4 = 0$$

may be given by

$$(m-2)^2 = 0.$$

Hence, we see that the auxiliary equation has a repeated root $m = 2$.

The general solution is therefore

$$y = (A + Bx)e^{2x}. \tag{5}$$

Suppose however, that we are given the initial conditions

$$y = 3 \text{ and } \frac{dy}{dx} = 2 \text{ at } x = 0.$$

We will now consider how to find the particular solution satisfying these conditions. First, we substitute $x = 0$ and $y = 3$ into equation 5. Since $e^0 = 1$, we get

$$3 = A \tag{6}$$

Next, we differentiate equation 5 with respect to $x$

$$\frac{dy}{dx} = Be^{2x} + 2(A + Bx)e^{2x}$$

and set $x = 0$, $\frac{dy}{dx} = 2$ to obtain

$$2 = B + 2(A) \tag{7}$$

We can substitute $A = 3$ from equation 6 into equation 7 to obtain

$$2 = B + 6$$

Hence,

$$B = -4$$

The particular solution to the original homogeneous equation satisfying the given boundary conditions is therefore

$$y = (3 - 4x)e^{2x}.$$

**Worked example. Case 3: $b^2 < 4ac$.**

Suppose that we have the following ODE: $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$

Note that the form of the equation indicates that $y$ is to be seen as a function of an independent variable $x$, and also that, since the equation is already homogeneous, the complementary function will be the general solution. Now the roots of the auxiliary equation

$$m^2 - 2m + 2 = 0$$

may be given by the quadratic formula,

$$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

At this stage, we see that $m = \alpha \pm i\beta$, hence $\alpha = 1$, $\beta = 1$.

The general solution is therefore

$$y = (A\cos(x) + B\sin(x))\, e^x \tag{8}$$

Suppose however, that we are given the initial conditions

$$y = 1 \text{ and } \frac{dy}{dx} = 5 \text{ at } x = 0$$

We will now consider how to find the particular solution satisfying these conditions. First, we substitute $x = 0$ and $y = 1$ into equation 8. Since $e^0 = 1$, $\cos(0) = 1$ and $\sin(0) = 0$. we get

$$1 = A \tag{9}$$

Next, we differentiate equation 8 with respect to $x$

$$\frac{dy}{dx} = A\cos(x)e^x - A\sin(x)e^x + B\sin(x)e^x + B\cos(x)e^x$$

and set $x = 0$, $\dfrac{dy}{dx} = 5$ to obtain:

$$5 = A + B \tag{10}$$

We can substitute $A = 1$ from equation 9 into equation 10 to obtain

$$5 = 1 + B$$

Hence,

$$B = 4$$

The particular solution to the original homogeneous equation satisfying the given boundary conditions is therefore

$$y = (\cos(x) + 4\sin(x))\, e^x$$

## 10.2 Finding the General Solution

**For an ODE in the form:** $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$.

For the above three worked examples, we assumed the homogeneous case,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \tag{1}$$

We must now consider the case whereby

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \tag{2}$$

Again, $a$, $b$ and $c$ are constants.

We have previously mentioned that the general solution, GS is given by: GS = CF + PI. In section 10.1, we have considered the various ways of finding CF - the solution of equation 1. We must now establish how to find PI, where again, PI is some function $y(x)$ that satisfies equation 2.

The basic method to solve such a problem and to find PI, is to use a trial solution which is essentially an initial guess. This trial solution will involve a number of parameters whose values can be determined by substituting into the complete equation.

For common functions, we may use the following suggested trial solutions:

| Case | Trial solution |
|---|---|
| $f(x) = p$ | $y = \frac{P}{C}$ |
| $f(x) = px + q$ | $y = Px + Q$ |
| $f(x) = px^2 + qx + r$ | $y = Px^2 + Qx + R$ |
| $f(x) = pe^{nx}$ | $y = Pe^{nx}$ |
| $f(x) = p\sin(nx)$ | $y = P\sin(nx) + Q\cos(nx)$ |
| $f(x) = p\cos(nx)$ | $y = P\sin(nx) + Q\cos(nx)$ |

Where $p$, $q$, $r$ and $n$ are given constants. $P$, $Q$ and $R$ are unknown constants, which may be determined by substituting into the original equation. Consider the worked example overleaf to understand the method.

**Worked example**

Suppose that we have the following ODE: $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2x^2$

Note that the form of the equation indicates that $y$ is to be seen as a function of an independent variable $x$. The homogeneous equation for this problem is simply given by:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Now the roots of the auxiliary equation

$$m^2 - 3m + 2 = 0$$

may be given by

$$(m-2)(m-1) = 0.$$

Hence, we see that the auxiliary equation has roots at $m = 1$ and $m = 2$.

The general solution is therefore

$$y = Ae^x + Be^{2x}$$

We now consider the particular solution for the differential equation. As $f(x) = 2x^2$, we begin by using a trial solution in the form $y = Px^2 + Qx + R$. From this, we have the following:

$$\begin{aligned} y &= Px^2 + Qx + R \\ y' &= 2Px + Q \\ y'' &= 2P \end{aligned}$$

So,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = y'' - 3y' + 2y = 2P - 3(2Px + Q) + 2(Px^2 + Qx + R)$$

Which may be rearranged to give:

$$y'' - 3y' + 2y = 2Px^2 + (2Q - 6P)x + (2R - 3Q + 2P)$$

Our ODE was given by: $y'' - 3y' + 2y = 2x^2$, therefore:

$$2Px^2 + (2Q - 6P)x + (2R - 3Q + 2P) = 2x^2$$

To find constants $P$, $Q$ and $R$, we now simply equate coefficients of $x^2$, $x$ and the constant term.

| Term | LHS coefficient | RHS coefficient | Conclusion |
|---|---|---|---|
| $x^2$ | $2P$ | $2$ | $P = 1$ |
| $x$ | $2Q - 6P$ | $0$ | $2Q - 6P = 0$<br>$2Q - 6 = 0$<br>$Q = 3$ |
| Constant | $2R - 3Q + 2P$ | $0$ | $2R - 3Q + 2P = 0$<br>$2R - 9 + 2 = 0$<br>$R = \frac{7}{2}$ |

Combining our results then, the general solution of the differential equation is given by:

$$y = x^2 + 3x + \frac{7}{2} + Ae^x + Be^{2x}$$

## Exercise

Solve the following using an auxiliary equation:

1. $\dfrac{d^2y}{dx^2} + 9\dfrac{dy}{dx} + 20y = 0$
2. $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 15y = 0$
3. $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 13x = 0$
4. $\dfrac{d^2x}{dt^2} + 16x = 0$
5. $\dfrac{d^2x}{dt^2} + 10\dfrac{dx}{dt} + 25x = 0$
6. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$
7. $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = t + 3e^{-t}$
8. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 2e^{x}$

# 11 Solutions

**Section 3:**

1. $y = \dfrac{x}{1 - Cx}$
2. $y = \pm\sqrt{2(x^2 + 3x + C)}$

**Section 4:**

1. $y = \dfrac{1}{2}\left(\pm k\sqrt{k^2 + 8x} - k^2 - 2x\right)$ Where $k = e^{\left(\frac{C}{2}\right)}$
2. $y = x(\tan(\ln(Cx)))$

**Section 5:**

1. $y = \pm\sqrt{C + 2\left(\dfrac{x^2}{2} + 5x\right) + (2x-1)^2} - 2x + 1$
2. $21x - 42y + 9\ln(14x + 21y + 22) = C$

**Section 6:**

1. $y = \dfrac{e^x}{2x} + \dfrac{C}{xe^x}$
2. $y = \dfrac{e^x + C}{x^3}$

**Section 7:**

1. $y = \dfrac{6Cx^{3/2} + 9C^2 + x^3}{9x}$
2. $y = \dfrac{1}{x^4(C - \ln(x))}$

**Section 8:**

1. $y = 2 + \dfrac{1}{-\dfrac{1}{3} + Ce^{(-3x)}}$ Hint: try $y = 2$
2. $y = \dfrac{2x}{C - x^2} + \dfrac{1}{x}$ Hint: try $y = \dfrac{1}{x}$

**Section 9:**

1. $-x^2\cos(2y) - y^4 - x = C$
2. $\pm\dfrac{\sqrt{C + 6x^3 + \frac{1}{2}(x^2+1)^2}}{\sqrt{2}} + \dfrac{1}{2}(-x^2 - 1)$

**Section 10:**

1. $Ae^{-5x} + Be^{-4x}$

2. $Ae^{-3x} + Be^{-5x}$

3. $(A\sin(3t) + B\cos(3t))e^{-2t}$

4. $A\sin(4t) + B\cos(4t)$

5. $(A + Bt)e^{-5t}$

6. $(A + Bx)e^{3x}$

7. $\left(A + Bt + \frac{3}{2}t^2\right)e^{-t} + t - 2$

8. $(A\sin(x) + B\cos(x))\,e^{-x} + \frac{2}{5}e^{x}$