Mathematics
Support Centre

Title: Laplace Transforms

Target: On completion of this worksheet you should be able to recognise and use Laplace transforms on standard functions. You should be able to recognise the properties of Laplace transforms as well as be able to use Laplace transforms to solve differential equations.

Contents

1 Introduction

The Laplace transform is an integral operator with a variety of applications in Mathematics, and other scientific areas. For instance, the Laplace transform is known to have applications in the following areas:

- Control engineering
- Communication
- Signal analysis and design
- Probability theory

In academics, Laplace transforms are commonly used to aid solving ordinary differential equations. This booklet outlines the definition of the Laplace transform and its properties, as well as explaining the process of using Laplace transforms to solve ordinary differential equations.

We may define the Laplace transform formally in the following way:

Suppose that $f(t)$ is a function defined for $0 \leq t < \infty$ and that s is a real positive parameter, then the Laplace transform of $f(t)$, denoted by $F(s) = \mathcal{L}\{f(t)\}$, is formally defined as:

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s)}$$

Essentially, to find the Laplace transform of a function you must multiply the function by e^{-st} and then integrate between 0 to ∞ with respect to t.

When studying Laplace transforms, you may encounter a wide variety of notation. Below are a few common examples of notation and their meaning:

1. $\tilde{f} \to F(s)$
2. $\dot{x} \to \dfrac{dx}{dt}$
3. $\ddot{x} \to \dfrac{d^2x}{dt^2}$
4. $f_0 \to f(0)$

2 Finding Laplace transforms

Below are three examples of finding Laplace transforms of functions, by using the formal definition.

Worked example

Suppose that we have the function $f(t) = 2$ for $t \geq 0$, and wish to find its Laplace transform.

Using the Laplace transform formula, we see that:

$$\mathcal{L}\{2\} = \int_0^\infty e^{-st}.2dt$$

By evaluating the integral, we obtain the following:

$$\begin{aligned}\mathcal{L}\{2\} &= 2\int_0^\infty e^{-st}dt \\ &= 2\left[\frac{e^{-st}}{-s}\right]_0^\infty \\ &= 2\left[0 - \left(\frac{1}{-s}\right)\right] \\ &= \frac{2}{s}\end{aligned}$$

Provided $s > 0$.

Note: This result may actually be generalised for $f(t) = k$, where k is any constant, to give

$$\mathcal{L}\{k\} = \frac{k}{s}$$

Again, provided $s > 0$.

Worked example

Suppose that we have the function $f(t) = e^{-kt}$ for $t \geq 0$, and wish to find its Laplace transform.

Using the Laplace transform formula, we see that:

$$\mathcal{L}\{e^{-kt}\} = \int_0^\infty e^{-st}.e^{-kt}dt$$

By evaluating the integral, we obtain the following:

$$\begin{aligned}\mathcal{L}\{e^{-kt}\} &= \int_0^\infty e^{-(s+k)t}dt \\ &= \left[\frac{e^{-(s+k)t}}{-(s+k)}\right]_0^\infty \\ &= \left[0 - \left(-\frac{1}{s+k}\right)\right] \\ &= \frac{1}{s+k}\end{aligned}$$

Provided $s + k > 0$.

Worked example

Suppose that we have the function $f(t) = \sin(at)$ for $t \geq 0$, and wish to find its Laplace transform.

To do this, we may choose to express $\sin(at)$ in complex exponential form. From the relationship,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

We see that $\sin(\theta)$ is simply the imaginary part of $e^{i\theta}$. We denote the imaginary part of $e^{i\theta}$ as $\Im(e^{i\theta})$. Using the Laplace transform formula, we see that:

$$\mathcal{L}\{\sin(at)\} = \mathcal{L}\{\Im(e^{iat})\} = \Im\int_0^\infty e^{-st}.e^{iat}dt$$

By evaluating the integral, we obtain the following:

$$\begin{aligned}
\mathcal{L}\{\Im(e^{iat})\} &= \Im\int_0^\infty e^{-st}.e^{iat}dt \\
&= \Im\int_0^\infty e^{-(s-ia)t}dt \\
&= \Im\left\{\left[\frac{e^{-(s-ia)t}}{-(s+ia)}\right]_0^\infty\right\} \\
&= \Im\left\{\left[0-\left(-\frac{1}{s-ia}\right)\right]\right\} \\
&= \Im\left\{\frac{1}{s-ia}\right\}
\end{aligned}$$

We have established then that $\mathcal{L}\{\sin(at)\} = \Im\left\{\frac{1}{s-ia}\right\}$

We may rationalise this result by multiplying the numerator and denominator by $s + ia$, this gives:

$$\mathcal{L}\{\sin(at)\} = \Im\left\{\frac{s+ia}{s^2+a^2}\right\}$$

By taking the imaginary part of the above result, we see that:

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

Note: The same method can be used to find $\mathcal{L}\{\cos(at)\}$, with the result given by

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$$

P.T.O.

For most common functions, you may wish to use the table below to quickly find various Laplace transforms.

Function $f(t)$	Laplace transform $\mathcal{L}\{f(t)\}$
K, constant	$\frac{K}{s}$
e^{-at}	$\frac{1}{s+a}$
t^n, where n is a positive integer	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sin(at) - at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$H(t-T)$, the Heaviside step function	$\frac{1}{s}e^{-sT}$
$\delta(t-T)$	e^{-sT}
$\frac{dx}{dt}$	$sX(s) - x_0$, where $x_0 = x(0)$
$\frac{d^2x}{dt^2}$	$s^2X(s) - sx_0 - x_1$, where $x_1 = \dot{x}(0)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$

3 Properties of Laplace transforms

We may now consider the various useful properties of the Laplace transform. By understanding these properties, we will develop important techniques that are essential when using Laplace transforms to solve differential equations.

3.1 Linearity

We begin by considering the linear property of Laplace transforms.

Let f and g be two functions of t. The linearity of Laplace transforms states that:

$$\boxed{\mathcal{L}\{f(t)+g(t)\}=\mathcal{L}\{f(t)\}+\mathcal{L}\{g(t)\}}$$

Essentially, the Laplace transform of a sum of functions is simply the sum of the Laplace transforms of the individual functions.

Furthermore, for any arbitrary constant k,

$$\mathcal{L}\{k.f(t)\}=k\mathcal{L}\{f(t)\}$$

This means that if we multiply a function by a constant k, then the corresponding transform is also multiplied by k.

Worked example

Suppose that we have the function $f(t)=3e^{-4t}+8\cos(5t)$ for $t\geq 0$, and wish to find its Laplace transform.

By the linearity property, we obtain:

$$\begin{aligned}\mathcal{L}\{3e^{-4t}+8\cos(5t)\}&=\mathcal{L}\{3e^{-4t}\}+\mathcal{L}\{8\cos(5t)\}\\&=3\mathcal{L}\{e^{-4t}\}+8\mathcal{L}\{\cos(5t)\}\end{aligned}$$

From the Laplace transform table given in section 2, we see that:

$$\begin{aligned}3\mathcal{L}\{e^{-4t}\}+8\mathcal{L}\{\cos(5t)\}&=3\left(\frac{1}{s+4}\right)+8\left(\frac{s}{s^2+5^2}\right)\\&=\frac{3}{s+4}+\frac{8s}{s^2+25}\end{aligned}$$

Hence,

$$\mathcal{L}\{3e^{-4t}+8\cos(5t)\}=\frac{3}{s+4}+\frac{8s}{s^2+25}$$

Exercise

Find the Laplace transforms of the following functions:

1. $f(t)=7t^3+2e^{-5t}$

2. $f(t)=5\cos(\pi t)+4t\sin(\pi t)$

3.2 First shift theorem

The first shift theorem states that if $\mathcal{L}\{f(t)\} = F(s)$ and k is any constant then the following property applies:

$$\boxed{\mathcal{L}\{e^{kt}f(t)\} = F(s-k)}$$

This property is referred to as the first shift theorem because transform function $F(s)$ is shifted a distance k along the s axis by the presence of the factor e^{kt}.

Worked example

Suppose that we have the function $f(t) = e^{-3t}\sin(2t)$ for $t \geq 0$, and wish to find its Laplace transform.

From the Laplace transform table given in section 2, we see that:

$$\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4} = F(s)$$

The first shift theorem states that:

$$\mathcal{L}\{e^{-3t}\sin(2t)\} = F(s - -3)$$

Meaning,

$$\mathcal{L}\{e^{-3t}\sin(2t)\} = F(s+3)$$

We have already established a result for $F(s)$, it follows from this that:

$$F(s+3) = \frac{2}{(s+3)^2+4}$$

Therefore,

$$\mathcal{L}\{e^{-3t}\sin(2t)\} = \frac{2}{(s+3)^2+4}$$

Exercise

Find the Laplace transforms of the following functions:

1. $f(t) = e^{5t}\cos(4t)$
2. $f(t) = e^{2t}t^8$

3.3 Multiplication by t^n

Another property of the Laplace transform states that if $\mathcal{L}\{f(t)\} = F(s)$ and n is a positive integer then

$$\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}}$$

Worked example

Suppose that we have the function $f(t) = t\cos(t)$ for $t \geq 0$, and wish to find its Laplace transform.

From the Laplace transform table given in section 2, we see that:

$$\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1} = F(s)$$

At this stage, we use the property $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$

We see that:

$$\mathcal{L}\{t\cos(t)\} = (-1)\frac{d}{ds}\left\{\frac{s}{s^2+1}\right\}$$

To finalise our solution, we must evaluate the derivative of $\frac{s}{s^2+1}$. This is given by:

$$\frac{d}{ds}\left\{\frac{s}{s^2+1}\right\} = \frac{d}{ds}\left\{s(s^2+1)^{-1}\right\}$$

By the differentiation product rule, this gives:

$$\begin{aligned}\frac{d}{ds}\left\{s(s^2+1)^{-1}\right\} &= (s^2+1)^{-1} - 2s^2(s^2+1)^{-2} \\ &= \frac{1}{s^2+1} - \frac{2s^2}{(s^2+1)^2}\end{aligned}$$

By combining fractions, this gives:

$$\begin{aligned}&= \frac{s^2+1}{(s^2+1)^2} - \frac{2s^2}{(s^2+1)^2} \\ &= \frac{1-s^2}{(s^2+1)^2}\end{aligned}$$

As $\mathcal{L}\{t\cos(t)\} = (-1)\frac{d}{ds}\left\{\frac{s}{s^2+1}\right\}$, we see that:

$$\mathcal{L}\{t\cos(t)\} = -\frac{1-s^2}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

Exercise

Find the Laplace transforms of the following functions:

1. $f(t) = t^3 e^{-3t}$

2. $f(t) = t^2 e^{-t}$

3.4 Inverse Laplace transforms

We have discussed how to obtain the Laplace transform, $F(s)$, for a given a function $f(t)$.
We must now consider how to attain a function $f(t)$, if given its Laplace transform $F(s)$. The property of inverse Laplace transforms may be summarised in the following way:

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = f(t)}$$

Worked example

Suppose that we need to invert the following Laplace transform $F(s) = \dfrac{s+1}{s(s^2-4)}$

First, we rewrite this as:

$$F(s) = \frac{s+1}{s(s+2)(s-2)}$$

By partial fractions, we see that this may be written as:

$$F(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-2}$$

Where A, B and C are unknown constants.

It can be shown that $A = -\dfrac{1}{4}$, $B = -\dfrac{1}{8}$ and $C = \dfrac{3}{8}$. From this then, we see that:

$$F(s) = -\frac{1}{4s} - \frac{1}{8(s+2)} + \frac{3}{8(s-2)}$$

We now take the inverse Laplace transform of the above, which we denote as $\mathcal{L}^{-1}\{F(s)\}$.

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{4s} - \frac{1}{8(s+2)} + \frac{3}{8(s-2)}\right\}$$

By the linearity property of Laplace transforms, this gives:

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{4s}\right\} + \mathcal{L}^{-1}\left\{-\frac{1}{8(s+2)}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{8(s-2)}\right\}$$

Which by the linearity property once more, gives:

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)}\right\} + \frac{3}{8}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\}$$

From the Laplace transform table given in section 2, we see that this gives:

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{4} - \frac{1}{8}\left(e^{-2t}\right) + \frac{3}{8}\left(e^{2t}\right)$$

Hence,

$$f(t) = -\frac{1}{4} - \frac{1}{8}\left(e^{-2t}\right) + \frac{3}{8}\left(e^{2t}\right)$$

Exercise

Find the inverse Laplace transforms of the following functions:

1. $F(s) = \dfrac{1}{s(s+1)}$
2. $F(s) = \dfrac{s+1}{s(s+2)}$

3.5 Second shift theorem

The second shift theorem states that if $\mathcal{L}\{f(t)\} = F(s)$ and a is any constant then the following property applies:

$$\boxed{\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s)}$$

Where $H(t-a)$ is the Heaviside step function, which is defined by the statements:

$$H(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

Worked example

Suppose that we have the function $(t-3)^2H(t-3)$ for $t \geq 0$, and wish to find its Laplace transform.

We see that $f(t-3) = (t-3)^2$, therefore, $f(t) = t^2$.

If $F(s)$ is given by $\mathcal{L}\{f(t)\}$, then from the Laplace transform table given in section 2, we see that:

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \\ &= \mathcal{L}\{t^2\} \\ &= \frac{2}{s^3} \end{aligned}$$

Referring back to the given formula,

$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s)$$

We see that:

$$\mathcal{L}\{H(t-3)f(t-3)\} = e^{-3s}F(s)$$

Hence,

$$\mathcal{L}\{H(t-3)(t-3)^2\} = \frac{2e^{-3s}}{s^3}$$

Note: this property may also be inverted to give:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)f(t-a)$$

Exercise

Find the Laplace transforms of the following functions:

1. $g(t) = (t-4)H(t-4)$

2. $g(t) = (t-7)^3H(t-7)$

3.6 Convolution theorem

For two functions of t, f and g we must recognise the following:

$$\boxed{\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}}$$

Essentially, this means that the Laplace transform of the product of two functions is *not* the same as the product of the Laplace transforms.
For $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = L\{g(t)\}$, consider some $H(s)$ such that $H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}$ for $s > a$. From this, the following property exists:

$$\boxed{h(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(t)g(t-\tau)d\tau}$$

Where $h(t)$ is known as the convolution of f and g and the integrals above are convolution integrals.

Worked example

Suppose that we have the function $h(t) = \int_0^t (t-\tau)\sin(2\tau)d\tau$, and wish to find its Laplace transform.

To solve this, we initially note that for $f(t) = t$ and $g(t) = \sin(2t)$, we obtain the following from the Laplace transform table given in section 2:

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}$$

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}$$

By the Convolution theorem then, we obtain the following result:

$$\mathcal{L}\{h(t)\} = H(s) = F(s)G(s) = \frac{2}{s^2(s^2+4)}$$

Exercise
Find the Laplace transforms of the following functions:

1. $h(t) = \int_0^t \sin(\tau)\sin(t-\tau)d\tau$

2. $h(t) = \int_0^t \tau sin(t-\tau)d\tau$

4 Using Laplace transforms to solve ODE's

Laplace transforms are a highly valuable tool when attempting to solve ordinary differential equations. Typically, the process of solving ODE's using Laplace transforms consists of four steps:

- Transform the entire equation
- Apply given boundary conditions
- Rearrange to make $X(s)$ the subject
- Find the inverse transform

Worked example

Suppose that we have an ODE in the form: $\ddot{x} + 3\dot{x} + 2x = 0$, $x(0) = 0$, $\dot{x}(0) = 1$.

As mentioned above, we begin by finding the Laplace transform of the entire equation. We have:

$$\mathcal{L}\{\ddot{x} + 3\dot{x} + 2x\} = \mathcal{L}\{0\}$$

By the linearity property, this becomes:

$$\mathcal{L}\{\ddot{x}\} + 3\mathcal{L}\{\dot{x}\} + 2\mathcal{L}\{x\} = \mathcal{L}\{0\}$$

From the Laplace transform table given in section 2, we see that:

$$\left(s^2 X(s) - sx_0 - \dot{x}_0\right) + 3\left(sX(s) - x_0\right) + 2X(s) = 0$$

We now apply the given boundary conditions, this gives:

$$\left(s^2 X(s) - 1\right) + 3\left(sX(s)\right) + 2X(s) = 0$$

We now rearrange the above to give:

$$X(s)\left(s^2 + 3s + 2\right) = 1$$

We now make $X(s)$ the subject, we obtain:

$$X(s) = \frac{1}{s^2 + 3s + 2}$$

This may be written as:

$$X(s) = \frac{1}{(s+2)(s+1)}$$

By partial fractions, this may be written as:

$$X(s) = \frac{A}{s+2} + \frac{B}{s+1}$$

It can be shown that $A = -1$ and $B = 1$. From this then, we see that:

$$X(s) = \frac{-1}{s+2} + \frac{1}{s+1}$$

We must now find the inverse Laplace transform, to find $x(t)$,

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+2} + \frac{1}{s+1}\right\}$$

By the linearity property, this becomes:

$$= -\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

From the Laplace transform table given in section 2, we see that this gives:

$$x(t) = -e^{-2t} + e^{-t}$$

Exercise

Solve the following ODE's:

1. $\ddot{x} + \dot{x} - 2x = 0$, $x(0) = 3$, $\dot{x}(0) = 0$

2. $\ddot{x} - x = e^{2t}$, $x(0) = 0$, $\dot{x}(0) = 1$

5 Solutions

Section 3.1 solutions

1. $F(s) = \dfrac{42}{s^4} + \dfrac{2}{s+5}$
2. $F(s) = \dfrac{5s}{s^2+\pi^2} + \dfrac{8\pi s}{(s^2+\pi^2)^2}$

Section 3.2 solutions

1. $F(s) = \dfrac{s-5}{(s-5)^2+16}$
2. $F(s) = \dfrac{8!}{(s-2)^9}$

Section 3.3 solutions

1. $F(s) = \dfrac{6}{(s+3)^4}$
2. $F(s) = \dfrac{2}{(s+1)^3}$

Section 3.4 solutions

1. $f(t) = 1 - e^{-t}$
2. $f(t) = \dfrac{1}{2} + \dfrac{1}{2}e^{-2t}$

Section 3.5 solutions

1. $\mathcal{L}\{(t-4)H(t-4)\} = \dfrac{e^{-4s}}{s^2}$
2. $\mathcal{L}\{(t-7)^3H(t-7)\} = \dfrac{6e^{-7s}}{s^4}$

Section 3.6 solutions

1. $\mathcal{L}h(t) = H(s) = F(s)G(s) = \dfrac{1}{(s^2+1)^2}$
2. $\mathcal{L}h(t) = H(s) = F(s)G(s) = \dfrac{1}{s^2} - \dfrac{1}{s^2+1}$

Section 4 solutions

1. $x(t) = 2e^t + e^{-2t}$
2. $x(t) = -\dfrac{1}{3}e^{-t} + \dfrac{1}{3}e^{2t}$