E1

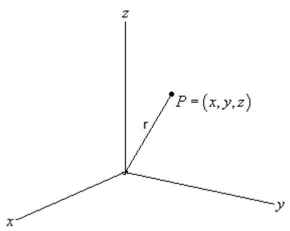
Mathematics Support Centre

Title: Position and displacement

Target: On completion of this worksheet you should be able to describe the position of physical objects

Reference frames

In classical mechanics, if we want to describe a point in space we need to use 3 **coordinates**. We traditionally name them x (left-right motion), y (forth-back) and z (up-down). On top of that we need to choose some **origin**, that is the point $O = (0,0,0)$, from which an observer measures the position of different objects. Together, this forms a **Euclidean coordinate system** assigned to an observer.



Position

As time progresses some objects change their position as measured by the observer. We call this phenomenon **motion**. For example, some object can have a description of

$$x(t) = 2\cos(\pi t)$$
$$y(t) = -10t^2$$
$$z(t) = 5$$

as measured by an observer. The coordinate z of the object does not change as time increases and hence the motion is 2 dimensional. With this description we can answer questions of the form: Where was the particle initially at $t = 0$? To find the answer we substitute $t = 0$ to the above formulae:

$$x(0) = 2\cos(\pi \times 0) = 2 \times 1 = 2$$
$$y(0) = -10 \times 0^2 = 0$$
$$z(0) = 5 = 5$$

An alternative way of describing position is by creating a **position vector**. Our object can be described in the vector notation as:
$\mathbf{r}(t) = 2\cos(\pi t)\hat{\mathbf{i}} - 10t^2\hat{\mathbf{j}} + 5\hat{\mathbf{k}} = (2\cos(\pi t), -10t^2, 5)$.

Displacement and distance

For any 2 points, $\mathbf{r_1} = (x_1, y_1, z_1)$, $\mathbf{r_2} = (x_2, y_2, z_2)$ we can define **displacement** - a vector from the initial to the final position of a body. It is given by $\mathbf{d} = \mathbf{r}_2 - \mathbf{r}_1$. The magnitude of displacement gives the length of the straight line connecting the 2 points. The Pythagoras theorem tells us that the straight-line distance l between two points is given by
$l = |\mathbf{r}_2 - \mathbf{r}_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
As an example, to find the distance of our object from the initial position at $t = 1$, we first find the corresponding position vectors
$\mathbf{r}(1) = (2\cos(\pi \times 1), -10 \times 1^2, 5) = (-2, -10, 5)$,
$\mathbf{r}(0) = (2, 0, 5)$ and then apply the Pythagoras theorem to get
$l = \sqrt{(-2-2)^2 + (-10-0)^2 + (5-5)^2} = \sqrt{116} \approx 10.77$ units.

Exercises

1. For a particle given by

$$x(t) = 5\sin(2\pi t)$$
$$y(t) = 0$$
$$z(t) = -10t$$

   find its position at: a) $t = 0$, b) $t = 1$, c) $t = \frac{1}{2}$. Is the motion 2 or 3 dimensional?

2. A body is prescribed by the position vector $\mathbf{r}(t) = (3\cos(\pi t), -3t^2, 3t - 1)$. Find the position vector of the particle at $t = 0$ and $t = 1$. What is the distance between those two points? Is the motion 1, 2 or 3 dimensional?

3. What is my displacement today if I woke up in my bed, went to MSC and came back to my bed? Is the distance I travelled the same as the displacement?

4. Prove that for any two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, $|\mathbf{r}_1 - \mathbf{r}_2| = |\mathbf{r}_2 - \mathbf{r}_1|$.

Answers

1.

$$\begin{aligned} a)\quad & x(0)=0 \\ & y(0)=0 \\ & z(0)=0 \\ b)\quad & x(1)=0 \\ & y(1)=0 \\ & z(1)=-10 \\ c)\quad & x(\tfrac{1}{2})=0 \\ & y(\tfrac{1}{2})=0 \\ & z(\tfrac{1}{2})=-5 \end{aligned}$$

   Alternatively, you can write the position vector. The motion is 2 Dimensional, since only y coordinate does not vary with time.

2. $r(0)=(3,0,-1),\quad r(1)=(-3,-3,2)$.
   $l=\sqrt{54}=3\sqrt{6}\approx 7.35$ units. The motion is 3 dimensional.

3. The displacement is 0, since at the end I came to the point I started from. The total distance I travelled is at least twice the distance from my bed to MSC, it depends on the route I took.

4. $l=|\mathbf{r}_2-\mathbf{r}_1|=$
   $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.
   But $(x_2-x_1)^2=x_2^2-2x_2x_1+x_1^2=(x_1-x_2)^2$.
   So
   $l=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}=|\mathbf{r}_2-\mathbf{r}_1|$.