E10

Mathematics Support Centre

Title: Work, Energy, Power

Target: On completion of this worksheet you should be able to apply the conservation of energy principle to physical systems

Work

Given a path $\mathbf{s}$, the work done by a force between the points a and b on that path is given by $W = \int_a^b \mathbf{F}\cdot \mathsf{d}\mathbf{s}$. If the force is constant and the motion is rectilinear, this further simplifies to $W = \mathbf{F}\cdot\mathbf{s} = |\mathbf{F}| l\cos(\theta)$, where θ is the angle between the force's direction and the path of length l. As an example, the work done by friction on a inclined plane is $W_f = \int_a^b \mathbf{F_f}\cdot \mathsf{d}s = \int_a^b \mu_k mg\cos(\alpha)\mathsf{d}s = \mu_k mg\cos(\alpha) l$, where μ_k is the coefficient of dynamic friction. The result follows from the fact that friction is always parallel to the path. Work is measured in Joules ($1\text{J} = 1\text{kg}\cdot\text{m}^2/\text{s}^2$).

Gravitational potential and kinetic energy

Energy is a measure of a body's capability of performing work. Here we focus on the 2 basic types of energy. For every non-dissipative force we can define an associated **potential energy** E_p. For example, The **gravitational potential energy** at a height h is equal to the negative of the work done in raising a body from the height h_0 to h. We therefore have

$$E_g = -\int_{h_0}^{h} \mathbf{F_g}\cdot \mathsf{d}\mathbf{s} = -\int_{h_0}^{h}(-mg)\mathsf{d}s = mg(h-h_0).$$

In practice, we are free to choose our coordinates so that $h_0 = 0$ to simplify it to $E_g = mgh$. Another example is the **elastic potential energy** which is equal to the negative of the work needed to stretch a string from rest position to x_0 meters. We have from the Hooke's law $F_s = -kx$,

$$E_s = -\int_0^{x_0}(-kx)\mathsf{d}x = \left.\frac{x^2}{2}\right|_0^{x_0} = \frac{x_0}{2}.$$

The second type of energy we're going to consider is the **kinetic energy**. It is equal to the work needed to accelerate a body from 0 to a speed u. If the body gains this speed between the points a and b we have

$$\begin{aligned}
E_k &= \int_a^b \mathbf{F}\cdot \mathsf{d}\mathbf{s} \\
&= \int_a^b m\mathbf{a}\cdot \mathsf{d}\mathbf{s}, \text{ since } \mathbf{F} = m\mathbf{a} \\
&= m\int_a^b \mathbf{a}\cdot \mathsf{d}\mathbf{s}, \text{ since } m = const. \\
&= m\int_a^b \frac{\mathsf{d}\mathbf{v}}{\mathsf{d}t}\cdot\frac{\mathsf{d}\mathbf{s}}{\mathsf{d}t}\mathsf{d}t, \text{ since } \mathbf{a} = \frac{\mathsf{d}\mathbf{v}}{\mathsf{d}t} \text{ and } \mathsf{d}\mathbf{s} = \frac{\mathsf{d}\mathbf{s}}{\mathsf{d}t}\mathsf{d}t \\
&= m\int_0^u \mathbf{v}\cdot \mathsf{d}\mathbf{v}, \text{ since } \frac{\mathsf{d}\mathbf{s}}{\mathsf{d}t} = \mathbf{v} \\
&= \left.\frac{m|\mathbf{v}|^2}{2}\right|_0^u = \frac{mu^2}{2}
\end{aligned}$$

The sum E_g, E_s and E_k is sometimes called the **total energy** of a system.

Conservation of energy

Along with the conservation of momentum, the second fundamental law of physics is that:
The total energy in an isolated system is conserved, that is, if no external forces are acting on a system, we have $E_k + E_p = const.$ The potential energy E_p has to be interpreted as E_g, E_s or their sum, accordingly to the systems nature. We also have an important consequence that if work of N Joules is done on a system, then its total energy increases by N Joules.

Power

The mean power is the the rate at which work is done. It is expressed as $\bar{P} = \frac{\Delta W}{\Delta t}$, where ΔW is the total work performed over the time Δt. The **instantaneous power** is the rate of change of work with respect to time $P = \frac{\mathsf{d}W}{\mathsf{d}t}$. Power is measured in Watts ($1\text{W}=1\text{J/s}=1\text{kg}\cdot\text{m}^2/\text{s}^3$).

Exercises

1. A device lifted a box of mass 10kg to 2m height in 2 seconds. Take $g = 9.81\text{m/s}^2$.
   a) What is the work done by the device?
   b) What is the mean power of the device?
   The box is then released and dropped to the ground.
   c) Using the conservation of energy find the speed of impact of the box with the ground.

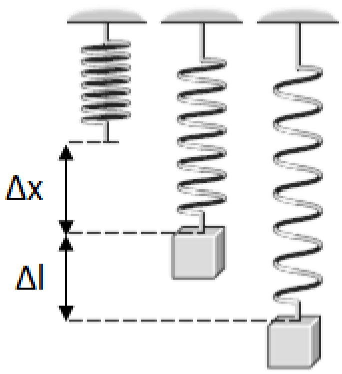
2. A pendulum of length $2m$ is released from an angle $\alpha = 30°$ with the vertical. What is the maximum speed of the bob?

3. Consider a spring-mass system satisfying the Hooke's law, namely $F = -kx$. The mass of the

body attached to the spring is m.
a) Calculate the work needed to stretch the spring by a distance x_0. What is the potential energy of the spring when stretched?
b) Find the maximum kinetic energy of the body if released from x_0.

4. Use the conservation of energy to solve:
A box is placed on a wooden ramp. The mass of the box is 2kg. The angle of the ramp with the horizontal is $\alpha = 30°$. The box slides down the wooden ramp freely for 2 meters. The dynamic friction coefficient is $\mu_k = 0.3$. Then, the box continues on a nearly frictionless surface of ice for 5 meters, to finally start climbing a 1 m long wooden ramp with the same μ_k, of the slope of $45°$.
a) Will the box jump the ramp or start sliding back? b) Compare your method with the one presented in question 4d) in the worksheet E6. Which method do you find easier?
c) How much energy was lost due to heat assuming it was the only source of dissipation?

5. A spring of a spring constant k is hung at the ceiling. Then a mass m is attached to it. The spring moves down and is stopped at a new equilibrium position at a distance Δx below.



a) What is the distance between the new and the previous equilibrium positions in terms of m, g and k?
b) The spring is then extended by an additional distance Δl downwards and released. What is the velocity at the equilibrium point? Take $m = 0.5$kg, $k = 2$N/m and $\Delta l = 0.2$m. Solve symbolically first and then put the values in your derived formula.

Answers

1. a) $W = mgh = 196.2J$
b) $\bar{P} = \frac{mgh}{\Delta t} = 98.1W$
c) $v = \sqrt{2gh} = 6.26$m/s

2. The bob raised by $l - l\cos(\theta) = 2l\sin(\frac{\theta}{2})^2 = 0.27$m. So the maximum speed is approximately 2.3m/s

3. a) $W = \displaystyle\int_0^{x_0} -kx\mathrm{d}x = -\frac{kx_0^2}{2}$, so $E_p = -W = \dfrac{kx_0^2}{2}$.
b) $v = x_0\sqrt{\frac{k}{m}}$

4. a) The potential energy at the top of the first ramp is $E_{p_1} = mgl_1\sin(\alpha_1) \approx 19.62$J. The potential energy at the top of the second ramp is $mgl_2\sin(\alpha_2) \approx 13.87$J. If there was no friction the box would have the kinetic energy of $E_k = \Delta E_p = 5.75$J as it leaves the second ramp due to the conservation of energy. The friction however on the entire track does the work of $W_f = \mu_k(mg\cos(\alpha_1)l_1 + mg\cos(\alpha_2)l_2) \approx 14.36$J. Since $W_f > \Delta E_p$ the box won't make the jump.
c) The energy dissipated in the process was $\Delta E_p \approx 5.75$J.

5. a) We have $F_g + F_s = 0$, i.e. $-mg + k\Delta x = 0$, $\Delta x = \frac{mg}{k}$
b) We take the 0 potential surface to be at the distance $\Delta x + \Delta l$ below the ceiling. Just before the mass is released it has the total energy of $E_t = E_k + E_g + E_s = 0 + 0 + \frac{k(\Delta x + \Delta t)^2}{2}$. At the equilibrium point it has the energy of $E_t = \frac{k(\Delta x)^2}{2} + mg\Delta l + \frac{mv^2}{2}$. Equating gives $\frac{k(\Delta x + \Delta t)^2}{2} = \frac{k(\Delta x)^2}{2} + mg\Delta l + \frac{mv^2}{2}$ and after simplifying we have $v = \Delta l\sqrt{\frac{k}{m}}$. Putting the values in results in $v = 0.4$m/s.