E2

Mathematics Support Centre

Title: Velocity and speed

Target: On completion of this worksheet you should be able to distinguish velocity from speed

(Instantaneous) Velocity

Velocity is a **vector**. As every vector, velocity has a magnitude and a direction. Given a body whose position is prescribed by the position vector $\mathbf{r}(t)$, the velocity is defined as the rate of change of position with respect to time, i.e. $\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}$. As an example consider a body described by the position vector $\mathbf{r}(t) = (\sin(\pi t), t^2, 5)$. The velocity vector of the body will be given by $\mathbf{v}(t) = (\pi\cos(\pi t), 2t, 0)$.

Integrating back to position

The task of getting the position vector from velocity is more complex. Let us try to integrate our example to see if we can recover the position. We see that $\mathbf{r}(t) = \int \mathbf{v}(t)\mathrm{d}t = \int (\pi\cos(\pi t), 2t, 0)\mathrm{d}t$. To integrate a vector we integrate every component individually to get $\mathbf{r}(t) = (\sin(\pi t) + C_1, t^2 + C_2, C_3)$, where the Cs are integration constants. To find the actual value of them we need to specify the **initial conditions**, that is the location of the particle at some initial time, usually taken to be $t = 0$. Now, given that $\mathbf{r}(0) = (0, 0, 5)$, we construct a system of equations to be solved for Cs:

$$\begin{aligned} \sin(\pi t)\big|_{t=0} + C_1 &= 0 \\ t^2\big|_{t=0} + C_2 &= 0 \\ C_3 &= 5 \end{aligned}$$

We get that $C_1 = 0$, $C_2 = 0$ and $C_3 = 5$, what finally results in $\mathbf{r}(t) = (\sin(\pi t), t^2, 5)$ as expected.

$$\mathbf{r}(t) \xmapsto{\frac{\mathrm{d}}{\mathrm{d}t}} \mathbf{v}(t)$$

$\mathbf{v}(t) \xmapsto{\int \ldots \mathrm{d}t} \mathbf{r}(t)$ (apply initial conditions)

Speed

Speed is the magnitude of velocity. It is therefore a **scalar**. We have $s(t) = |\mathbf{v}(t)| = \sqrt{v_x^2 + v_y^2 + v_z^2}$, where $v_x = \frac{\mathrm{d}x(t)}{\mathrm{d}t}, v_y = \frac{\mathrm{d}y(t)}{\mathrm{d}t}, v_z = \frac{\mathrm{d}z(t)}{\mathrm{d}t}$. Returning to our example, the speed of the body is $s(t) = \sqrt{(\pi\cos(\pi t))^2 + (2t)^2 + 0^2} = \sqrt{\pi^2\cos(\pi t)^2 + 4t^2}$.

Average values

The **average velocity** ($\bar{\mathbf{v}}$) is the total displacement ($\Delta\mathbf{r}$) divided by the total time of the motion (Δt). It is a vector. We have $\bar{\mathbf{v}} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r}_{\text{fin}} - \mathbf{r}_{\text{init}}}{\Delta t}$.

The **average speed** ($\bar{s}$) is the total distance travelled (Δl) divided by the total time of the motion. It is a scalar. Therefore $\bar{s} = \frac{\Delta l}{\Delta t}$

Exercises

1. Find the corresponding velocity vector for motion described by:

$$a) \begin{cases} x(t) &= t\sin(t) \\ y(t) &= \cos(3t) \\ z(t) &= \sqrt{t^2+1} \end{cases} \quad b)\ \mathbf{r}(t) = (t^3+1, \sin(2t)) \quad c)\ \mathbf{r}(t) = \cosh(0.5t)\hat{\mathbf{i}}$$

2. Knowing that $\mathbf{r}(0) = (1, 1, 1)$, find the position vector from $\mathbf{v}(t) = (\pi\cos(\pi t), t^2 + 1, \exp(5t))$

3. A particle follows a 2D motion given by

$$\begin{aligned} x(t) &= \cos(2\pi t) \\ y(t) &= \sin(2\pi t) \end{aligned}$$

   a) Write down the position vector.
   b) Find the velocity vector.
   c) Find the speed of the particle.
   d) The particle is stopped after 3 seconds of motion. What are the average speed and average velocity?
   e) What physical situations can you think of that exhibit this behaviour?

4. In an experiment, a basketball is dropped from a h meters tall building. The students know that before it reaches the ground, the velocity equation for the motion of the basketball is $v(t) = -\sqrt{\frac{g}{k}}\tanh\left(\frac{t}{\tau}\right)$, where g, k and τ are know constants. Given that $\frac{\mathrm{d}\ln(\cosh(t))}{\mathrm{d}t} = \tanh(t)$ find the position vector of the ball as a function of time while it's mid-air.

5. Prove that for any motion, the magnitude of the average velocity, $|\bar{\mathbf{v}}|$, is always less or equal to the average speed, $\bar{s}$.

Answers

1. Since position differentiates to velocity we have:

   a) $v_x(t) = \sin(t) + t\cos(t)$ (product rule)
   $v_y(t) = -3\sin(3t)$
   $v_z(t) = \frac{t}{\sqrt{t^2+1}}$ (chain rule)

   b) $\mathbf{v}(t) = (3t^2, 2\cos(2t))$
   c) $\mathbf{v}(t) = 0.5\sinh(0.5t)\hat{\mathbf{i}}$

2. $\mathbf{r}(t) =$
   $= \left(\sin(\pi t) + C_1, \frac{1}{3}t^3 + t + C_2, \frac{1}{5}\exp(5t) + C_3\right)$.
   Applying the initial conditions results in

   $$C_1 = 1$$
   $$C_2 = 1$$
   $$\frac{1}{5} + C_3 = 1 \Rightarrow C_3 = \frac{4}{5}$$

   Finally $\mathbf{r}(t) =$
   $= \left(\sin(\pi t) + 1, \frac{1}{3}t^3 + t + 1, \frac{1}{5}\exp(5t) + \frac{4}{5}\right)$

3. a) $\mathbf{r}(t) = (\cos(2\pi t), \sin(2\pi t))$ or
   $\mathbf{r}(t) = \cos(2\pi t)\hat{\mathbf{i}} + \sin(2\pi t)\hat{\mathbf{j}}$
   b) $\mathbf{v}(t) = (-2\pi\sin(2\pi t), 2\pi\cos(2\pi t))$
   c) $s(t) = \sqrt{(-2\pi\sin(2\pi t))^2 + (2\pi\cos(2\pi t))^2} = 2\pi$ d) $\bar{s} = 2\pi$, since the speed is constant. $\bar{v} = 0$, since $\mathbf{r}(3) - \mathbf{r}(0) = 0$ e) Those are the equations for **uniform circular motion**. An electron in a magnetic field behaves that way, the Earth orbits the Sun in a nearly circular orbit, etc.

4. $r(t) = h - \tau\sqrt{\frac{g}{k}}\ln(\cosh(\frac{t}{\tau}))$.

5. Assume that $|\bar{\mathbf{v}}| > \bar{s}$. Then from the definition we have

   $$\left|\frac{\Delta\mathbf{r}}{\Delta t}\right| > \frac{\Delta l}{\Delta t}$$
   $$\frac{|\Delta\mathbf{r}|}{|\Delta t|} > \frac{\Delta l}{\Delta t}$$
   $$\frac{|\Delta\mathbf{r}|}{\Delta t} > \frac{\Delta l}{\Delta t}$$
   $$|\Delta\mathbf{r}| > \Delta l$$

   Since $|\Delta r|$ is the length of the straight line connecting the start and end points of the motion, it is the smallest distance between the two points. Δl cannot be smaller than $|\Delta r|$. Therefore we proved that $|\bar{\mathbf{v}}| \leq \bar{s}$. The equality $|\bar{\mathbf{v}}| = \bar{s}$ holds if and only if the motion is **rectlinear**, i.e. it follows a straight line. It also needs to have a constant direction.