E3

**Mathematics Support Centre**

# **Title:** Acceleration

**Target:** On completion of this worksheet you should be able to calculate acceleration

### (Instantaneous) Acceleration

Acceleration is a **vector**. It is defined as the rate of change of velocity with respect to time, that is $\mathbf{a}(t) = \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2}$. For example, given a velocity vector $\mathbf{v}(t) = (\pi\cos(\pi t), 2t, 0)$, the acceleration will be $\mathbf{a}(t) = (-\pi^2\sin(\pi t), 2, 0)$. What follows from the definition is that the acceleration can also be acquired from differentiating position twice.

### Position-Velocity-Acceleration

The following diagram gives the relation between the 3 vectors used to describe motion

$$\mathbf{r}(t) \xmapsto{\frac{\mathrm{d}}{\mathrm{d}t}} \mathbf{v}(t) \xmapsto{\frac{\mathrm{d}}{\mathrm{d}t}} \mathbf{a}(t)$$

$$\mathbf{a}(t) \xmapsto{\int\ldots\mathrm{d}t} \mathbf{v}(t) \xmapsto{\int\ldots\mathrm{d}t} \mathbf{r}(t)$$

As we can see, to get the velocity vector, we need to integrate acceleration. As an example consider $\mathbf{a}(t) = (3\exp(-5t), \sin(\pi t), 3t^2)$. Velocity will then be given by $\mathbf{v}(t) = \int\mathbf{a}(t)\mathrm{d}t = (-\frac{3}{5}\exp(-5t) + C_1, -\frac{1}{\pi}\cos(\pi t) + C_2, t^3 + C_3)$, where the $C$s are constants of integration. To find the values of the constants we need specified **initial conditions** for velocity. For example, given that at $t = 0$,
$\mathbf{v}(0) = (-\frac{3}{5}, -\frac{1}{\pi}, 1)$ we get:

$$-\frac{3}{5}\exp(-5t)\big|_{t=0} + C_1 = -\frac{3}{5}$$
$$-\frac{1}{\pi}\cos(\pi t)\big|_{t=0} + C_2 = -\frac{1}{\pi}$$
$$t^3\big|_{t=0} + C_3 = 1$$

So, $C_1 = 0$, $C_2 = 0$, and $C_3 = 1$. We finally get that $\mathbf{v}(t) = (-\frac{3}{5}\exp(-5t), -\frac{1}{\pi}\cos(\pi t), t^3 + 1)$. Applying the same principles, we can now integrate the calculated velocity to get the position vector. For instance take that at $t = 3$, $\mathbf{r}(3) = (1, 1, 1)$. We get $\mathbf{r}(t) = (\frac{3}{25}\exp(-5t) + D_1, -\frac{1}{\pi^2}\sin(\pi t) + D_2, \frac{1}{4}t^4 + t + D_3)$, where $D$s are constants. Applying the initial conditions results in

$$\frac{3}{25}\exp(-5t)\big|_{t=3} + D_1 = 1$$
$$-\frac{1}{\pi}\cos(\pi t)\big|_{t=3} + D_2 = 1$$
$$(\frac{1}{4}t^4 + t)\big|_{t=3} + D_3 = 1$$

Since $\frac{3}{25}\exp(-15) \approx 0$ we have $D_1 \approx 1$, $D_2 = 1 - \frac{1}{\pi}$, $D_3 = -\frac{89}{4}$. That results in $\mathbf{r}(t) =$
$= (\frac{3}{25}\exp(-5t) + 1, -\frac{1}{\pi^2}\sin(\pi t) + 1 - \frac{1}{\pi}, \frac{1}{4}t^4 + t - \frac{89}{4})$.
Notice, that the initial conditions do not have to be taken at $t = 0$.

### Average acceleration

Sometimes we are interested in measuring the average value of the acceleration of a certain body. It is defined by $\bar{\mathbf{a}} = \frac{\Delta\mathbf{v}}{\Delta t}$, where $\Delta\mathbf{v} = \mathbf{v}_{fin} - \mathbf{v}_{init}$ is the change of velocity and $\Delta t$ is the elapsed time. Average acceleration is therefore a **vector**.

### Exercises

1. Find the acceleration of a body given that:
   a) its velocity is $\mathbf{v}(t) = (5\exp(-2t), \sin(\pi t))$.
   b) its position is $\mathbf{r}(t) = \sin(2t)\hat{\mathbf{i}} + \dfrac{3}{2t^2}\hat{\mathbf{j}}$.
   c) $\begin{cases} x(t) &= \exp(-t^2) \\ y(t) &= \sin(t)\cos(t) \\ z(t) &= 2t + 1 \end{cases}$

   Find the value of acceleration at t=0 for all the cases.

2. Find the position of a body whose acceleration is $\mathbf{a}(t) = (5, \sin(2\pi t), \exp(-2t)))$. Initially, the body is at the origin and has the initial velocity of $\mathbf{v}(0) = (1, -\frac{1}{2\pi}, -\frac{1}{2})$. What is the position at $t = 2$ s?

3. A car accelerates uniformly from 17.6 m/s to 48.2 m/s in 3.76 seconds. Find the acceleration of the car and the distance travelled.

4. Derive the general formula for velocity and position of a body subject to a constant acceleration. The body is initially at $r(0) = r_0$ and $v(0) = v_0$.

5. A car is moving at a constant speed of 30 m/s on a straight line. The motion lasts for 3.76s. What is the acceleration of the car?

**Answers**

1. a) $\mathbf{a}(t) = (-10\exp(-2t), \pi\cos(\pi t))$ and $\mathbf{a}(0) = (-10, \pi)$
   b) $\mathbf{a}(t) = -4\sin(2t)\hat{\mathbf{i}} + \dfrac{9}{t^4}\hat{\mathbf{j}}$ and $\mathbf{a}(t)$ is undefined at $t = 0$ due to division by 0.
   c) Hints: Use the chain rule for $x(t)$ to get $v_x(t)$. Use the product rule along with the chain rule to get $a_x(t)$. Use trigonometric identities for $y(t)$ to get $v_y(t)$.
   $\mathbf{a}(t) = (\exp(-t^2)(4t^2 - 2), -4\sin(t)\cos(t), 0)$ and $\mathbf{a}(0) = (-2, 0, 0)$

2. $\mathbf{v}(t) =$
   $(5t + 1, -\frac{1}{2\pi}\cos(2\pi t), -\frac{1}{2}\exp(-2t))$,
   $\mathbf{r}(t) =$
   $(\frac{5t^2}{2} + t, -\frac{1}{4\pi^2}\sin(2\pi t), \frac{1}{4}(\exp(-2t) - 1))$.
   $\mathbf{r}(2) \approx (12, 0, -0.25)$

3. $\Delta v = 30.6$ m/s. $\Delta t = 3.76$ s. $a = \frac{\Delta v}{\Delta t} \approx 8.14$ m/s$^2$. $v(t) = 8.14t + 17.6$. $r(t) = 4.07t^2 + 17.6t + r_0$. Since we are only interested in the total distance the car travelled we get $\Delta r(t) = 4.07t^2 + 17.6t$. So $\Delta r(3.76) = 123.7$ m.

4. $a(t) = const. = a$.
   $v(t) = at + v_0$.
   $r(t) = \frac{1}{2}at^2 + v_0 t + r_0$.

5. The acceleration is 0, since the speed is constant and the motion is rectlinear.