E9

Mathematics Support Centre

Title: Momentum and Impulse

Target: On completion of this worksheet you should be able to apply momentum conservation law to physical systems.

Momentum and its conservation law

Momentum is defined to be the product of mass and velocity of an object. It is therefore given by $\mathbf{p} = m\mathbf{v}$. The unit of momentum is 1kg·m/s. From Newton's second law we know that, for a body with a constant mass, we have

$$\begin{aligned} m\mathbf{a} &= \mathbf{F} \\ m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \mathbf{F} \\ \frac{\mathrm{d}m\mathbf{v}}{\mathrm{d}t} &= \mathbf{F}, \text{ since mass is constant} \\ \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= \mathbf{F}. \end{aligned} \quad (1)$$

If $\mathbf{F} = 0$, this further reduces to $\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = 0$, or

$$\mathbf{p} = \begin{pmatrix} C_x \\ C_y \\ C_z \end{pmatrix}, \text{ where the } Cs \text{ are constants.} \quad (2)$$

Equation (1) is the Newton's second law expressed in the momentum form. It turns out that it works for both objects with a constant mass analysed earlier, but also for objects whose mass changes with time, i.e. $m(t) \neq const$. Equation (2) is the **principle of conservation of momentum**, a fundamental concept in physics, which can be stated as:
The total momentum of an isolated system remains constant regardless of changes within the system.

Impulse

Due to the fundamental theorem of calculus, the equation (1) can be rewritten as

$$\Delta\mathbf{p} = \int_{t_i}^{t_f} \mathbf{F}\mathrm{d}t = \mathbf{J}$$

$\Delta\mathbf{p}$ is the **change of momentum** of a body which was acted upon by a force of $\mathbf{F}$ during the time $t_f - t_i = \Delta t$. J is the **impulse**. When the force is constant or Δt is very small, we can simplify the equation to .

$$\Delta\mathbf{p} = \mathbf{F}\Delta t$$

Applications in merging and splitting systems

An immediate consequence of the conservation of momentum is that if an isolated system is split into n bodies, then the sum of their momenta is constant. We have $\mathbf{p}_{total} = \underbrace{\mathbf{p}_1 + \mathbf{p}_2 + ... + \mathbf{p}_n}_{n \text{ times}} = \sum_{i=1}^{n} \mathbf{p}_i = \sum_{i=1}^{n} m_i\mathbf{v}_i = const$. This equality holds at any time. For example let us consider the following: Two boys are sailing in a boat. The total weight of the boat including the boys is 100kg. They are moving at 1m/s. Suddenly the boys decide to jump off the boat, one through the bow (front) and one through the stern (back) of the boat. The boys weigh 30kg and 40kg, respectively, and jump at the speed of 0.5m/s each with respect to the boat. What is the final speed of the boat after they jump?
To answer this question we firstly notice that the momenta in the system is conserved since it is isolated. Initially the boats momentum was $p_i = mv_0 = 100$kg·m/s. After the system split, we have that $p_f = \sum_{i=1}^{n} m_i v_i = \overbrace{30 \times (1 + 0.5)}^{\text{'bow boy'}} + \overbrace{40 \times (1 - 0.5)}^{\text{'stern boy'}} + \overbrace{30v_b}^{\text{boat}}$. Equating $p_i = p_f$ and rearranging gives $v_b \approx 1.17$m/s.

Exercises

1. A girl weighting 45kg is riding a skateboard at 1.3m/s carrying a stone of mass 5kg. What is her final speed after she throws the stone with the speed of 0.4m/s, as seen by her:
   a) straight ahead?
   b) vertically down?
   c) straight back?

2. A canon of mass 2500kg is placed on ice and fires a cannonball of mass 20kg at an angle $\frac{\pi}{6}$ radians with the horizontal. The projectiles initial speed was 1080km/h. The time during which the force was acting on the projectile was 1s.
   a) Calculate the impulse that acted on the cannonball.
   b) Calculate the force that acted on the projectile.
   c) Calculate the speed of the cannon after it fired.

3. a) A particle of mass m_p hits a wall at speed v_p and sticks to it. What is the change of the momentum of the particle?
   b) A particle of mass m_p hits a wall at speed v_p and bounces of it at the same speed. What is the change of the momentum of the particle?
   c) The MESSENGER probe orbiting Mercury used a device called the **solar sail** for orbit corrections during its mission. It's a device that works on the quantum phenomena of assigning momentum p_γ to individual particles of light. One side of the sail (black) absorbs the light almost completely, while the other (white) reflects it. Based on your answers in 3a) and 3b) explain what causes the sail to move, when an equal amount of particles hits it from both sides.

Answers

1. a) 1.26m/s, b) 1.3m/s, c) 1.34m/s
2. a) Since $\Delta\mathbf{p} = \mathbf{J}$ we have $\mathbf{J} = 300 \times 20 = 6000$N·s or equivalently 6000kg·m/s.
   b) It follows that $\mathbf{F} = \frac{\mathbf{J}}{s} = 6000$N.
   c) From the conservation of the momenta (x component), we have $v \approx 2.08$m/s.
3. a) $\Delta p = m_p v_p$, b) $\Delta p = 2m_p v_p$
   c) Given $2N$ particles hitting both sides of the sail, we get that the net change in momentum is $\Delta p = Np_\gamma$. Therefore, the sail will have to accelerate so as not to violate Newton's second law.