E8

**Mathematics Support Centre**

# Title: Newton's Laws of Motion and Differential Equations

**Target:** On completion of this worksheet you should be able to derive equations of motion of a physical system

### Force

We return to the concept of force which was already covered in the worksheet E4. Some forces are just expressed as a constant. They are independent of time and position, for example, gravity $F_g = mg$. Other forces change with time, for example, $F(t) = \sin(2\pi t)$ or with position, like the Hooke's law $F_s(x) = kx$. During motion, the position of a body changes with time. That's why we can write the Hooke's law as $F_s(x(t)) = kx(t)$ to underline time dependency of the force. In classical mechanics, we mostly focus on the forces of the form $\mathbf{F} = \mathbf{F}(t, \mathbf{x}(t), \mathbf{v}(t), \mathbf{a}(t))$, which depend on time, position of the body and the time derivatives of position (velocity and acceleration).

### Newton's second law for modelling motion

In the most general form, we have $\sum \mathbf{F} = m\mathbf{a}(t)$. Depending on the nature of the force, this is used to derive the equations of motion for a physical object. For example, for a damped harmonic oscillator, the forces acting on the body are the Hooke's force $F_s = -kx(t)$, and the linear dampening term $F_d = -cv(t)$, where we wrote $x = x(t), v = v(t)$ to indicate, that the quantities depend on time. The minus sign in $F_s$ and $F_d$ signifies the direction of the forces. This leads to:

$$\begin{aligned} F_s + F_d &= ma(t) \\ -kx(t) - cv(t) &= ma(t) \\ m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + c\frac{\mathrm{d}x(t)}{\mathrm{d}t} + kx(t) &= 0. \end{aligned}$$

This is a **linear differential equation** and it can be solved to find $x(t)$. In the case when the motion is 2D or 3D we need to solve equations of motion for each component of $x(t), y(t)$ and $z(t)$. Consider an electron moving in a uniform (constant) magnetic field perpendicular to its velocity. The force acting on it can be summarized as

$$\begin{cases} \mu\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} \\ \mu\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} \end{cases}$$

for some constant $\mu$.

### Deriving Newton's first law

Can we get an understanding of this law by starting from the second law? Let us analyse what happens when $\sum \mathbf{F} = 0$. We acquire

$$\begin{aligned} m\mathbf{a}(t) &= 0 \\ \mathbf{a}(t) &= 0 \qquad \text{(for a non-zero mass)} \\ \mathbf{v}(t) &= \mathbf{v_0} \end{aligned}$$

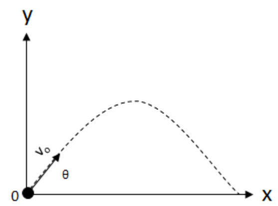
Therefore, we see that the 1st law follows naturally from the 2nd law, i.e. the body will be moving in a **uniform rectlinear motion** or will stay at rest if $\mathbf{v_0} = 0$.

### Exercises

1. A projectile fired at the **origin** at the angle $\alpha$ with an initial speed of $v_0$ will satisfy the system of equations

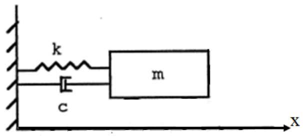
$$\begin{cases} 0 = ma_x(t) \\ -mg = ma_y(t). \end{cases}$$

   Solve the system of equations by integrating it twice and applying the right **initial conditions**. By eliminating time from the equation of $y(t)$, show that the trajectory is parabolic.




2. The differential equation governing the motion of a simple pendulum is $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\sin(\theta) = 0$, where $l$ is the length of the pendulum, $g$ the gravity constant and $\theta$ the angular displacement from the equilibrium position.
   a) Derive this equation from Newton's second law.
   b) The derived equation is non-linear and difficult to solve, but, if the angle $\theta$ is small ($\theta < 4°$), we can use the following approximation: $\sin(\theta) \approx \theta$. Use this approximation to acquire a linearised equation for the system.
   c) Find the general solution of the newly derived equation.

3. Find the general solution for a damped oscillator in 1D that is **not** excited by any external forces. What happens when the dampening constant is 0?



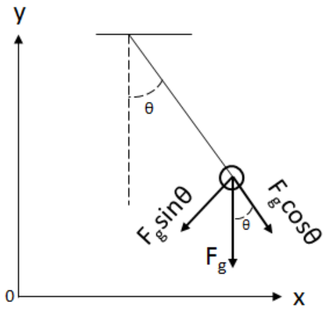

4. A spherical body in free fall in a high-viscosity medium is affected by a drag force $F_d = -cv(t)$. Find the equation of motion in terms of velocity and solve it using the separation of variables.

**Answers**

1.

$$
\begin{aligned}
v_x(t) &= v_0 \cos(\alpha) \\
v_y(t) &= -gt + v_0 \sin(\alpha) \\
x(t) &= v_0 \cos(\alpha) t \\
y(t) &= -\frac{gt^2}{2} + v_0 \sin(\alpha) t \\
y(x) &= -\frac{gx^2}{2v_0^2 \cos^2(\alpha)} + \tan(\alpha) x
\end{aligned}
$$

2. The system:




a) The force tangential to the bob is $-mg\sin(\theta)$. We have:

$$
\begin{aligned}
ma(t) &= -mg\sin(\theta(t)) \\
m\frac{\mathrm{d}^2 r(t)}{\mathrm{d}t^2} &= -mg\sin(\theta(t)) \\
\frac{\mathrm{d}^2 r(t)}{\mathrm{d}t^2} &= -g\sin(\theta(t)) \\
\frac{\mathrm{d}^2 l\theta(t)}{\mathrm{d}t^2} &= -g\sin(\theta(t)) \text{ as } r = l\theta(t) \\
l\frac{\mathrm{d}^2 \theta(t)}{\mathrm{d}t^2} &= -g\sin(\theta(t)) \text{ as } l \in \mathbb{R}. \\
\frac{\mathrm{d}^2 \theta(t)}{\mathrm{d}t^2} &+ \frac{g}{l}\sin(\theta(t)) = 0
\end{aligned}
$$

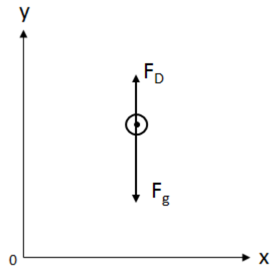
b) $\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + \frac{g}{l}\theta(t) = 0$

c) The characteristic polynomial is $\lambda^2 + \frac{g}{l} = 0$. That leads to $\lambda = \pm i\sqrt{\frac{g}{l}}$.

What follows is that $x(t) = A\cos(\sqrt{\frac{g}{l}}t) + B\sin(\sqrt{\frac{g}{l}}t)$ or $x(t) = C\sin(\sqrt{\frac{g}{l}}t + \phi_0)$, where $C = \sqrt{A^2 + B^2}$ and $\tan(\phi_0) = \frac{B}{A}$.

3. We have $m\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + c\frac{\mathrm{d}x(t)}{\mathrm{d}t} + kx(t) = 0.$, so the characteristic polynomial becomes: $m\lambda^2 + c\lambda + k = 0$. That equation is solved by $\lambda_1 = \frac{-c+\sqrt{c^2-4mk}}{2m}$ and $\lambda_2 = \frac{-c+\sqrt{c^2-4mk}}{2m}$. So finally $x(t) = A\exp(\lambda_1 t) + B\exp(\lambda_2 t)$. When $c = 0$ we recover the simple harmonic motion with the angular frequency of $\omega = \sqrt{\frac{k}{m}}$.

4. The system:




We have the gravity force $F_g = mg$ and the drag force $F_d = -cv$. That results in

$$
\begin{aligned}
ma &= F_g + F_d \\
ma &= mg - cv \\
\frac{\mathrm{d}v}{\mathrm{d}t} &= g - \frac{c}{m}v \\
\mathrm{d}t &= \frac{1}{g - \frac{c}{m}v}\mathrm{d}v \\
t &= \int \frac{1}{g - \frac{c}{m}v}\mathrm{d}v = -\frac{m}{c}\ln\left(g - \frac{c}{m}v\right) + C \\
v(t) &= \frac{gm}{c} - \left(\frac{gm}{c} - v(0)\right)\exp\left(-\frac{c}{m}t\right)
\end{aligned}
$$