S14

Mathematics Support Centre

Title: Poisson Distribution

Target: On completion of this worksheet you should be able to identify key assumptions and calcuate probabilites using the poisson distribution

Key Information to start:
The poisson distribution provides a good model for data that represents the number of occurrences of a specified event in a given time or space and calculates a probability.

Key Notation
$x =$ represents the number of events that occur in a period of time or space.
$\mu =$ The average number of times that an event occurs in a period of time or space.
$k =$ occurance of this event
$e =$ exponential component $= 2.7183...$

General form
We say that k follows a poisson distribution:
$$X \sim Po(\mu)$$
and we use the equation:
$$P(X = k) = \frac{\mu^k e^{-\mu}}{k!}$$

Key Assumptions
1) The events occur independently
2) The events occur at random
3) The probabilty of an event occurring in a given time interval does not vary with time

More Information
Since μ is the average number of successes occurring in a given time interval or region in a Poisson distribution then the mean and variance of Poisson is μ. Therefore standard deviation is $\sqrt{\mu}$.
In the Binomial distribution, if n is significantly high ($n >$ ...)and p is significantly small ($p <$ ...)(check with your lecture for when its too high or small for your exam/ coursework, as they can have different opinions on when its too significant), then we can approximate it to become a Poisson distribution by
$$\mu = np$$
This is because the values that you put in for the Binomial is too high that the calculator will give Math Error(i.e try ${}^{1000}C_{250}$). The way around it would be change it into Poission from above to get a very close approximation for large values of Binomial.

Example 1
The mean number of calls to a switchboard is 8 per hour. Assuming a poisson model, calculate the probability of exactly 5 calls during the next hour.

Answer
$X \sim Po(\mu)$ where $\mu = 8$ and $k = 5$
Using the formula
$$P(X = k) = \frac{\mu^k e^{-\mu}}{k!}$$
$$P(X = 5) = \frac{8^5 e^{-8}}{5!} = 0.092 = 9.2\%$$
Therefore there is a 9.2% chance of 5 phone calls during the next hour.

Example 2

The expected number of arrivals to a rollercoaster queue is 6 people every 5 minutes. Assuming a poisson distribution model, calculate the probability of:

a) exactly 7 arrivals in 10 minutes

b) more than 2 arrivals in the next 3 minutes

Answer

a) $X \sim Po(6)$ is for 6 people every 5 minutes. We need to change it to match 10 minutes. Therefore we manipulate μ and we need to change X to Y because of the new μ

Y = number of arrivals in 10 minutes so we double μ which is currently 6

$Y \sim Po(12)$

Using formula when μ is 12 and k is 7

$P(Y = 7) = \dfrac{12^7 e^{-12}}{7!} = 0.0437 = 4.37\%$

Probability of 7 arrivals in 10 minutes is 4.37%

b) Z = number of arrivals in 3 minutes so we times μ with $\dfrac{3}{5}$ to get $Z \sim Po(6 \times \dfrac{3}{5})$

using the formula where μ is 3.6 and k is greater than 2

$P(Z > 2) = 1 - P(Z \leq 2)$

$= 1 - (P(Z = 0) + P(Z = 1) + P(Z = 2))$

$$= 1 - (\frac{3.6^0 e^{-3.6}}{0!} + \frac{3.6^1 e^{-3.6}}{1!} + \frac{3.6^2 e^{-3.6}}{2!})$$

$= 0.697$

Please note: We use different letters to seperate our working but the formulas remain the same. Each letter represents the time frame for that expectance, but when all changed to the same time frame, they are all the same value.

Exercises

1. The number x of people entering the intensive care unit at a particular hospital on any one day follows a poisson distribution, with mean equal to 5 people per day.
   a) What is the probability that the number of people entering the intensive care unit on a particular day is 2?
   b) Less than or equal to 2?
   c) Is it likely that the number exceeds 10? Explain?

2. According to a study conducted by the Department of Pediatrics at the University of California, San Francisco, children who are injured 2 or more times tend to sustain these given injuries during a relatively limited time, usually 1 year or less. If the average number of injuries per year for school-age children is 2, what are the probabilities of these events?
   a) A school-age child will sustain 2 injuries during the year
   b) A school-age child will sustain at most one injury during the year.
   c) A school-age child will sustain 2 or more injuries during the year

3. If the probability that an individual will suffer a bad reaction from injection of a given Serum is 0.001 determine the probabilty that out of 2000 individuals
   a) exactly
   b) more than 2
   will suffer a bad reation

Answers

1. a) 0.08422

   b) 0.12465

   c) It is unlikely to exceed 10 because if you create a bar graph for each value, as it reaches 10 the probability decreases and becomes significantly small and anything above 10 has a very low probability. Also the mean is 5 and probability will be higher around 5 and since 10 is doubled, above 10 will be much lower value for probability.

2. a) 0.271

   b) 0.406

   c) 0.594

3. a) $p = 0.001, \quad n = 2000 \quad np = 2$
   $X \sim Po(2)$
   $= 0.18044$
   b) 0.3233